S. S. College. Jehanabad (Magadh University)

Department : Physics

Subject : Quantum Mechanics

Class : B.Sc(H) Physics Part III

Topic: Commutation Relation and Eigenvalue

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Commutator

The commutator of the operators A and B, denoted as $[A, B]$ and defined as

$$[A, B] = AB - BA$$

It follows that

$$[B, A] = -[A, B]$$

If $[A, B] = 0$ then $AB = BA \Rightarrow$ operator A and B commute. If $AB \neq BA$, operators does not commute.

Q: find the value of $\left[x, \frac{d}{dx}\right]$

Solution: we have for any function $\psi(x)$

$$\left[x, \frac{d}{dx}\right]\psi(x) = \left(x\frac{d}{dx} - \frac{d}{dx}x\right)\psi(x)$$

$$= x\frac{d\psi}{dx} - \frac{d}{dx}(x\psi(x)) = x\frac{d\psi}{dx} - x\frac{d\psi}{dx} - \psi(x)$$

$\left[x, \frac{d}{dx}\right]\psi(x) = -\psi(x)$, since $\psi(x)$ is arbitrary

$$\left[x, \frac{d}{dx}\right] = -1$$

Q Prove that

(i) $[A, B+C] = [A, B] + [A, C]$

$$[A, B+C] = A(B+C) - (B+C)A$$

$$= AB + AC - BA - CA$$

$$= AB - BA + AC - CA$$

$$= [A, B] + [A, C]$$

(ii) $[A, BC] = [A, B]C + B[A, C]$

$$\begin{aligned}[A, BC] &= ABC - BCA \\ &= ABC - BCA + BAC - BAC \\ &= ABC - BAC + BAC - BCA \\ &= (AB - BA)C + B(AC - CA) \\ &= [A, B]C + B[A, C]\end{aligned}$$

(iii) Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

$$[A, [B, C]] = [A, BC - CB] = A(BC - CB) - (BC - CB)A$$

$$[B, [C, A]] = [B, CA - AC] = B(CA - AC) - (CA - AC)B$$

$$[C, [A, B]] = [C, AB - BA] = C(AB - BA) - (AB - BA)C$$

Add All, we get

$$\begin{aligned}&= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA} + \cancel{BCA} - \cancel{BAC} - \cancel{CAB} + \\ &\quad \cancel{ACB} + \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC} \\ &= 0\end{aligned}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Eigen values and Eigen functions

If the result of applying an operator $\hat{A}$ on a wave function ψ is a scalar multiple of ψ itself

$$\hat{A}\psi = \lambda\psi$$

where α is a complex number, then ψ is called an eigen function of $\hat{A}$ and α is called the corresponding eigen value.

If there is whole set of eigenfunction ψ_n of an operator A, Then

$$\boxed{\hat{A}\psi_n = \alpha_n \psi_n}$$ eigenvalue equation

Degeneracy

If more than one linearly independent eigen-function have the same eigenvalue, then this eigen value is said to be degenerate.

Prove that :-

The eigen values of Hermitian operator are real.

$$\hat{A}\psi = \alpha \psi$$

where A is hermitian

$$\langle \psi | \hat{A} | \psi \rangle = \langle \hat{A}\psi | \psi \rangle$$

$$\alpha \langle \psi | \psi \rangle = \langle \alpha \psi | \psi \rangle$$

$$\alpha \langle \psi | \psi \rangle = \alpha^* \langle \psi | \psi \rangle$$

$$(\alpha - \alpha^*) \langle \psi | \psi \rangle = 0$$

$$\boxed{\alpha^* = \alpha}$$