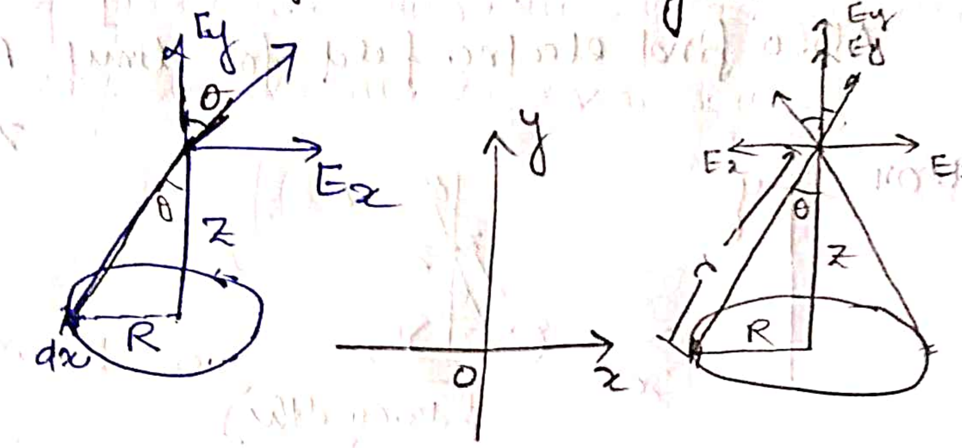
Problem: Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.

Solution



Here in figure electric field due to small length on circumference of circle is shown.
After dissolving Electric field and taking into consideration of symmetry the x component of electric field will be cancelled so only y component of electric field will contribute to net electric field due to circular loop.
So electric field due to small portion dx of loop

$$dE = \frac{dq}{4\pi\epsilon_0 r^2} \qquad [r^2 = R^2 + z^2]$$

here $dq = \lambda dx$ [λ charge per unit length]

y component of electric field

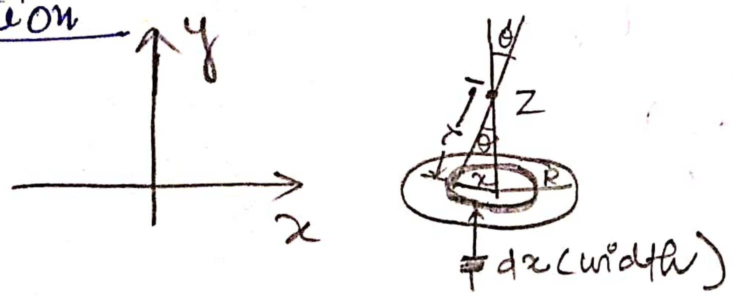
$$dE_y = dE\cos\theta$$

$$dE_y = \frac{\lambda dx}{4\pi\epsilon_0 r^2} \cdot \frac{z}{r} = \frac{z\lambda dx}{4\pi\epsilon_0 r^3}$$

$$\int dE_y = \int \frac{z\lambda dx}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} = \frac{\lambda z}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} \cdot 2\pi R$$

Problem: Find the electric field at a distance z above the center of a flat circular disk of radius R, which carries a uniform surface charge σ.
Also find electric field for limit: $R \to \infty$ and $z >> R$.

Solution



In disc we choose a small circle of radius x with thickness dx, producing electric field dE. Due to symmetry x component of ~~dE~~ electric field will be cancelled and y components will be added to produce net electric field.

$$dE_y = dE \cos\theta$$

$$dE = \frac{dq}{4\pi\epsilon_0 r^2}$$

$$\cos\theta = \frac{z}{r} \quad ; \quad r^2 = z^2 + x^2$$

$$dq = \sigma(2\pi x)\,dx$$

$$dE_y = \frac{\sigma\, 2\pi x\, dx}{4\pi\epsilon_0 (z^2+x^2)} \cdot \frac{z}{(z^2+x^2)^{1/2}}$$

$$E_y = \int dE_y = \int_{x=0}^{x=R} \frac{\sigma z}{2\epsilon_0} \frac{x\,dx}{(x^2+z^2)^{3/2}}$$

$$E_y = \int_{x=0}^{R} \frac{\sigma z}{2\epsilon_0} \frac{x\,dx}{(x^2+z^2)^{3/2}}$$

To solve

let $x = z\tan\theta$

$dx = z\sec^2\theta\, d\theta$

at $x = 0$ $\quad \tan\theta = 0$

$\theta = 0°$

at $x = R$ $\quad \tan\theta = R/z$

$\theta = \tan^{-1}(R/z)$

$$E_y = \frac{\sigma z}{2\epsilon_0} \int_{\theta=0}^{\tan^{-1}(R/z)} \frac{x\,dx}{(x^2+z^2)^{3/2}}$$

$$E_y = \frac{\sigma z}{2\epsilon_0} \int_{\theta=0}^{\tan^{-1}(R/z)} \frac{z\tan\theta \cdot z\sec^2\theta\, d\theta}{(z^2\tan^2\theta + z^2)^{3/2}}$$

$$E_y = \frac{\sigma z}{2\epsilon_0} \int_{\theta=0}^{\tan^{-1}(R/z)} \frac{z^2\tan\theta\sec^2\theta\, d\theta}{z^3 \cdot \sec^3\theta}$$

$$E_y = \frac{\sigma z}{2\epsilon_0} \times \frac{1}{z} \int_{\theta=0}^{\tan^{-1}(R/z)} \tan\theta \cdot \cos\theta\, d\theta$$

$$E_y = \frac{\sigma}{2\epsilon_0} \int_{\theta=0}^{\tan^{-1}(R/z)} \sin\theta\, d\theta$$

$$E_y = \frac{\sigma}{2\epsilon_0}\left[-\cos\theta\right|_{\theta=0}^{\tan^{-1}(R/z)}$$

$$E_y = \frac{\sigma}{2\epsilon_0}\left[-\cos\left(\tan^{-1}(R/z)\right) + \cos 0^\circ\right]$$

$$E_y = \frac{\sigma}{2\epsilon_0}\left[-\cos\left(\tan^{-1}(R/z)\right) + 1\right]$$

Let $\tan^{-1}(R/z) = \phi$

$\Rightarrow \quad R/z = \tan\phi$

$\Rightarrow \quad R^2/z^2 = \tan^2\phi$

$\Rightarrow \quad 1 + R^2/z^2 = 1 + \tan^2\phi$

$\Rightarrow \quad \dfrac{z^2 + R^2}{z^2} = \sec^2\phi$

$\Rightarrow \quad \sqrt{\dfrac{z^2+R^2}{z^2}} = \sec\phi$

$\Rightarrow \quad \cos\phi = \dfrac{z}{\sqrt{R^2+z^2}}$

$$E_y = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$$

Net Electric field $\quad E = \dfrac{\sigma}{2\epsilon_0}\left(1 - \dfrac{z}{\sqrt{R^2+z^2}}\right)$