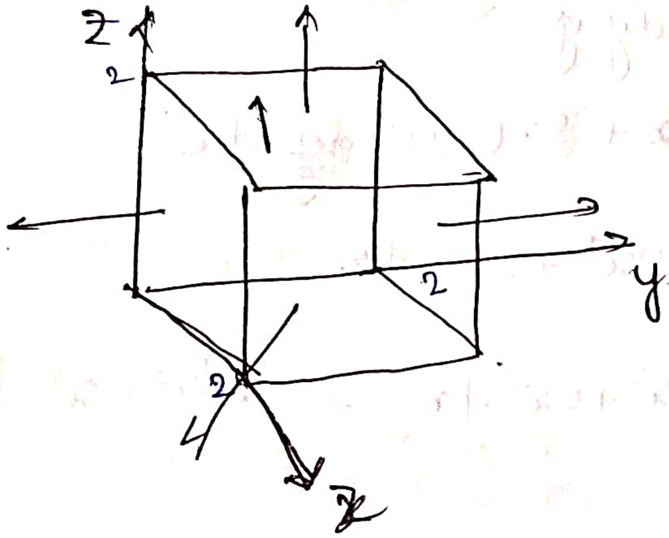
Q. Calculate the surface integral of
$\vec{v} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$ over five sides (excluding the bottom) of the cubical box. Let upward and outward be the positive direction as indicated by arrow




Taking the sides one at a time

(i) $x=2$ $\quad d\vec{a} = dy\,dz\,\hat{x}$

$$\vec{v}\cdot d\vec{a} = 2xz\,dy\,dz$$

$$= 4z\,dy\,dz$$

$$\int \vec{v}\cdot d\vec{a} = 4\int_0^2 dy \int_0^2 z\,dz = 4\int_0^2 dy \times 2$$

$$= 4\times 2\times 2 = 16$$

(ii) $x = 0$ $\quad \vec{da} = -dy\,dz\,\hat{x}$

$\vec{v}\cdot\vec{da} = -2xz\,dy\,dz = 0$

$\therefore \int \vec{v}\cdot\vec{da} = 0$

(iii) $y = 2$ $\quad \vec{da} = dx\,dz\,\hat{y}$

$\vec{v}\cdot\vec{da} = (x+2)\,dx\,dz$

$$\int \vec{v}\cdot\vec{da} = \int_{x=0}^{2}\int_{z=0}^{2} (x+2)\,dx\,dz$$

$$= \int_{x=0}^{2} (x+2)\,dx \int_{z=0}^{2} dz$$

$$= 2\left[\frac{x^2}{2} + 2x\right]_0^2 = 12$$

(iv) $y = 0$ $\quad \vec{da} = -dx\,dz\,\hat{y}$

$\vec{v}\cdot\vec{da} = -(x+2)\,dx\,dz$

$$\int \vec{v}\cdot\vec{da} = -\int_{x=0}^{2}\int_{z=0}^{2} (x+2)\,dx\,dz = -12$$

(v) $z = 2$, $\quad \vec{da} = dx\,dy\,\hat{z}$

$\vec{v}\cdot\vec{da} = y(z^2-3)\,dx\,dy$

$= y\,dx\,dy$ [Putting $z=2$]

$$\int \vec{v}\cdot\vec{da} = \int_{y=0}^{2}\int_{x=0}^{2} y\,dy\,dx = 2\int_{y=0}^{2} y\,dy = 4$$

~~Eq~~

Evidently the total flux is

$$\int_{surface} \vec{v} \cdot d\vec{a} = 16 + 0 + 12 - 12 + 4 = 20$$

Fundamental Theorem of Divergences.

The fundamental theorem for divergence states that:

$$\int_V (\nabla \cdot v)\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

This theorem also known as Gauss' theorem Green's theorem or simply divergence theorem

This theorem states that the integral of a derivative over a region is equal to the value of the function at the boundary (in this case the surface that bounds the volume).