Name of College – S.S. College, J-Bad
Deptt – Mathematics
TOPIC – problem (contd) on Definite Integration as limit of a sum.
Class – B.Sc I (Hons) Time – 10.15 A.M to 11.00 A.M
Date – 15-09-2020 11.00 A.M to 11.45 A.M

By – Dr. Shree Nath Sharma

Evaluate the following by summation

$\int_a^b \cos x\, dx$

Here $f(x) = \cos x \quad nh = b-a$

$f(a+rh) = \cos(a+rh)$

$$\int_a^b \cos x\, dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} \cos(a+rh)$$

$$= \lim_{h\to 0} h\left[\cos a + \cos(a+h) + \cos(a+2h) + \cdots + \text{up to } n \text{ terms}\right]$$

$$= \lim_{h\to 0} h \left\{\frac{\sin nh/2}{\sin h/2} \cos\left(a + \frac{n-1}{2}h\right)\right\}$$

$$= \lim_{h\to 0} h \cdot \frac{\sin\frac{(b-a)}{2}}{\frac{\sin h/2}{h/2} \times \frac{h}{2}} \cos\left[a + \frac{nh-h}{2}\right]$$

$$= \lim_{h\to 0} 2\sin\frac{(b-a)}{2} \cdot \cos\left[a + \frac{nh-h}{2}\right]$$

$$= 2\sin\frac{(b-a)}{2} \cos\left[a + \frac{b-a-0}{2}\right]$$

$$= 2\sin\frac{b-a}{2} \cos\left[\frac{2a+b-a}{2}\right]$$

$$= 2\sin\frac{b-a}{2} \cos\frac{(a+b)}{2}$$

$$= 2\cos\frac{a+b}{2} \sin\frac{b-a}{2}$$

$$= \sin b - \sin a$$

$\left[2\cos A \sin B = \sin(A+B) - \sin(A-B) \right.$

2. $\int_a^b e^x \, dx$

Here $f(x) = e^x \quad f(a+rh) = e^{a+rh}$

$= e^a \cdot e^{rh}$

$$\int_a^b e^x \, dx = \lim_{h \to 0} h \sum f(a+rh)$$

$$= \lim_{h \to 0} h \left[f(a) + f(a+h) + f(a+2h) + \dots \right] \text{ n terms}$$

$$= \lim_{h \to 0} h \left[e^a + e^a e^h + e^a e^{2h} + \dots \right] \text{ n terms}$$

$$= \lim_{h \to 0} e^a \cdot h \left[1 + e^h + (e^h)^2 + \dots \right]$$

$$= \lim_{h \to 0} e^a \cdot h \, \frac{(e^h)^n - 1}{e^h - 1} \qquad \left| \; a + ar + ar^2 + \dots = a \cdot \frac{r^n - 1}{r - 1} \text{ when } r > 1 \right.$$

$$= \lim_{h \to 0} e^a \, h \cdot \frac{e^{nh} - 1}{e^h - 1}$$

$\because nh = b - a$

$$= \lim_{h \to 0} e^a \cdot h \, \frac{e^{b-a} - 1}{1 + h + \frac{h^2}{2!} + \dots - 1}$$

$$= \lim_{h \to 0} e^a \, \cancel{h} \, \frac{e^{b-a} - 1}{\cancel{h} \left[1 + \frac{h}{2!} + \dots \right]}$$

$$= e^a \left(e^{b-a} - 1 \right)$$

$$= e^b - e^a.$$

$\int_0^1 x^3 dx$

Here $f(x) = x^3$

$a=0 \quad b=1 \quad nh = b-a = 1-0 = 1$

$f(a+rh) = f(0+rh) = r^3h^3.$

Now $\int_0^1 x^3 dx = \lim_{h\to 0} h \sum_{r=1}^{n} r^3 h^3$

$= \lim_{h\to 0} h^4 \frac{n^2(n+1)^2}{4}$ $\qquad \because \sum r^3 = \frac{n^2(n+1)^2}{4}$

$= \lim_{h\to 0} \frac{n^2h^2}{4} (nh+h)^2$

$= \frac{1}{4}(1+0)^2$

$= \frac{1}{4}$

<u>Limits of a sum as definite integral</u>

We know

$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$ or $\lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$

where $nh = b-a$

<u>Working rule</u> → STEP I find the rth term of the given series

STEP II – See whether $\frac{1}{n}$ is a factor; if not make $\frac{1}{n}$ a factor and obtain the given series in the form

$\lim_{n\to\infty} \sum \frac{1}{n} f(r/n)$

Put dx in place of $\frac{1}{n}$, x in place of r/n and then $\int$ in place of of $\sum$

Find the lower limit and upper limit of any the integral as

lower limit of the definite integral

= value of r/n for the first term as $n \to \infty$

upper limit of the definite integral

= value of r/n for the last term when $n \to \infty$

then when $a = 0$ $b = 1$ $h = 1/n$

we get $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \frac{1}{n} f(r/n) = \int_0^1 f(x)\,dx$

or

$$\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} f(r/n) = \int_0^1 f(x)\,dx.$$

Ex:→

Evaluate $\lim\limits_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \right]$

$$= \lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} \right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n+r}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{\infty} \frac{1}{n(1+r/n)}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{\infty} \frac{1}{1+r/n} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x}\,dx = \Big[\log(1+x)\Big]_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2 \quad \because \log 1 = 0$$

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{3n}\right]$$

$T_1 = n \qquad T_n = 3n$

<u>Now</u> No of term $= \dfrac{T_n - T_1}{d} = \dfrac{3n-n}{1} = 2n$

$T_r = n+(r-1)\times 1$

$\quad = n+r-1$

$$= \lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n+r}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n(1+r/n)}$$

a = Lower limit $= \lim_{n\to\infty} r/n$ when $r=0$

$\quad = 0$

b = upper limit $= \lim_{n\to\infty}\dfrac{2n}{n} = 2.$

$$= \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{(1+r/n)}\,\frac{1}{n}.$$

$$= \int_0^2 \frac{1}{1+x}\,dx = \Big[\log(1+x)\Big]_0^2$$

$$= \log 3 - \log 1$$

$$= \log 3 \qquad \because \log 1 = 0$$

Ans 5

Evaluate -

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$$

The Given Series can be written as

$$\lim_{n\to\infty}\left[\frac{n^2}{n^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{n^2}{(2n)^3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n^2}{n^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{n^2}{(n+n)^3}\right]$$

$\therefore$ (r+1) th Term $=\dfrac{n^2}{(n+r)^3}$ $\therefore$ the Given Series $\lim_{n\to\infty}\sum_{r=0}^{n}\dfrac{n^2}{(n+r)^3}$

$$=\frac{n^2}{n^3(1+r/n)^3}$$

$$=\frac{1}{n(1+r/n)^3}$$

$a=0$ when $r=0$ $a=0$

when $r=n$ $b=1$

$$\therefore \lim_{n\to\infty}\sum\frac{n^2}{(n+r)^3}=\lim_{n\to\infty}\sum\frac{1}{n(1+r/n)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}=\int_0^1(1+x)^{-3}\,dx$$

$$=-\left[\frac{1}{2(1+x)^2}\right]_0^1 \qquad =\left[\frac{(1+x)^{-2}}{-2}\right]_0^1$$

$$= -\left[\frac{1}{8} - \frac{1}{2}\right]$$

$$= -\left[\frac{1-4}{8}\right] = (-)(-3/8) = 3/8$$

Evaluate $\lim_{n\to\infty} \frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \dots + \frac{n^2}{n^3+n^3}$

Here rth term $= \frac{r^2}{r^3+n^3}$

$\therefore$ the given series $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{r^3+n^3}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3(1+r^3/n^3)} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2/n^2}{n(1+\frac{r^3}{n^3})}$$

$$a = \lim_{n\to\infty} \frac{1}{n^3} = 0$$

$$b = \lim_{n\to\infty} \frac{n^3}{n^3} = 1$$

$$= \int_{a=0}^{b=1} \frac{x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\int_{a=0}^{b=1} \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\left[\log 2 - \log 1\right] = \frac{1}{3}\log 2$$