Name of college - S.S. College, J-Bad.
Deptt - Mathematics
Topic - ~~Concept of~~ Study Material of Matrix (Contd)
Class - B.Sc (Hons + Sub)
Time - 1:10 P.M to 1:45 P.M
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Contd Suppose that the Matrices A, B and C are so ~~choosen~~ chosen that the matrix Multiplications are defined

(1) Then $(AB)C = A(BC)$ i.e Multiplication is associative

(2) for $k \in R$, $(kA)B = k(AB) = A(kB)$

(3) Then $A(B+C) = AB + AC$ i.e Multiplication distributes over Addition

(4) If A is $n \times n$ Matrix then
$$AI_n = I_n A = A.$$

(5) for any square Matrix A of order n and $D = \text{diag}(d_1, d_2, \cdots, d_n)$, we have

(i) The First Row of DA is d_1 times of the first Row of A

(ii) $1 \le i \le n$, the i^{th} row of DA is d_i times the ...

A Similar Statement holds for the co
of A when A is multiplied on the right

Ex:→ Let A and B be two Matrices. If the Matrix addition $A+B$ is defined, then prove that

$$(A+B)' = A' + B'$$

Also if the Matrix Product AB is defined then Prove that

$$(AB)' = B'A'$$

* A Matrix A over R is called Symmetric if $A = A'$

$\Rightarrow A - A' = 0$ is the Necessary condition for a Matrix be Symmetric

and if $A = A'$ $\Rightarrow$ Matrix A is Skew Symmetric

2. Matrix A is called orthogonal if $AA' = A'A = I$

E:→ Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix}$ is Symmetric Matrix

Since $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix} = A$

(ii) $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}_{3\times 3}$ is Skew Symmetric

Here $B' = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}_{3\times 3}$

$\Rightarrow -B' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

Here we see $B = -B'$ i.e $B + B' = 0$

(iii) If $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}_{3\times 3}$

then show that A is orthogonal Matrix

Since $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$

then $A' = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$

$$AA' = \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{2}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & 0 & -\frac{2}{\sqrt6} \end{bmatrix}$$

$C_{11} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$. $C_{12} = \frac{1}{\sqrt6} - \frac{1}{\sqrt2} + 0 = 0$ $C_{13} = \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}}$
$= 0$

$C_{21} = \frac{1}{\sqrt6} - \frac{1}{\sqrt6} + 0 = 0$ $C_{22} = \frac{1}{2} + \frac{1}{2} + 0 = 1$ $C_{23} = \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} +$
$= 0$

$C_{31} = \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} = 0$ $C_{32} = \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} + 0 = 0$ $C_{33} = \frac{1}{6} + \frac{1}{6} + \frac{4}{6}$
$= 1$

$$\therefore AA' = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly we may show

$$A'A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $AA' = A'A = I_3$

* Let. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Then $A^2 = A$.

$$A \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A.$$

Thus $A^2 = A$

The Matrices that satisfy the condition that $A^2 = A$ are called Idempotent Matrices.

* Show that for any square ~~Matrices~~ Matrix A

$S = \frac{1}{2}[A + A']$ is Symmetric.

$T = \frac{1}{2}(A - A')$ is skew Symmetric

and $A = S + T$

Say $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\therefore$ $A + A' = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$

$A - A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

thus $S = \frac{1}{2}(A + A'] = \frac{1}{2}\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}$

$S' = \begin{bmatrix} 1 & 5/4 \\ 5/2 & 4 \end{bmatrix}$

Here $S = S' \Rightarrow S = \frac{1}{2}(A + A')$ is Symmetric

$$T = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$T' = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = T$$

$$\cancel{T - T' = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}}$$

thus $T = \frac{1}{2}(A - A')$ is skew symmetric

Now $S + T = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$$= \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

Hence the required result.

* <u>SubMatrix of a Matrix</u>

Definition – A Matrix obtained by deleting some of the rows and or columns of a matrix is said to be a sub-matrix of the given matrix.

for example

if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$

then a few sub-matrices of A are

$[1]$ $[2]$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$

But the matrices

$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$

are not sub-matrices of A.

Block Matrices

Let A be an $n \times m$ matrix and B be an $m \times p$ matrix. Suppose $r < m$, then, we can decompose the matrices A and B as

$A = [P \; Q]$ and

$B = \begin{bmatrix} H \\ K \end{bmatrix}$, where P has order $n \times r$ and H has order $r \times p$. That is, the matrices P and Q are sub-matrices of A and P consists of the first r columns of A and Q consists of the last $m-r$ columns of A.

Similarly H and K are submatrices of B and H consists of the first r rows of B and B consists of the last $m-r$ rows of B. We now prove the following important theorem.

Theorem :→ Let $A = [a_{ij}] = [P\ Q]$ and $B = [b_{ij}] = \begin{bmatrix} H \\ K \end{bmatrix}$ be defined as above. Then

$$AB = PH + QK$$

Proof :→ First note that the matrices PH and QK are each of order $n \times p$. The matrix products PH and QK are valid as the order of the matrices P, H, Q and K are respectively, $n \times r$, $r \times p$, $n \times (m-r)$ and $(m-r) \times p$

Let $P = [p_{ij}]$ $Q = [q_{ij}]$ $H = [H_{ij}]$ and $Q = [q_{ij}]$ $H = [H_{ij}]$ and $K = [k_{ij}]$

then for $1 \leq i \leq n$ and $1 \leq j \leq p$

we have

$$(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{k=1}^{r} a_{ik} b_{kj} + \sum_{k=r+1}^{m} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{r} p_{ik} H_{kj} + \sum_{k=r+1}^{m} q_{ik} K_{kj}$$

$$= (PH)_{ij} + (QK)_{ij} = (PH + QK)_{ij}$$

Ex:→ if $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}_{3\times 2}$

then $AB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix}$

$$= \begin{bmatrix} a+2c & b+2d \\ 2a+5c & 2b+5d \end{bmatrix}$$

If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \\ -2 & 5 & -3 \end{bmatrix}$, then A can be decomposed as follows

$$A = \left[\begin{array}{c|cc} 0 & -1 & 2 \\ 3 & 1 & 4 \\ \hline -2 & 5 & -3 \end{array}\right] \quad \text{or} \quad A = \left[\begin{array}{cc|c} 0 & -1 & 2 \\ 3 & 1 & 4 \\ \hline -2 & 5 & -3 \end{array}\right] \quad \text{or}$$

$$A = \left[\begin{array}{cc|c} 0 & -1 & 2 \\ \hline 3 & 1 & 4 \\ -2 & 5 & -3 \end{array}\right]$$ and so on.

Suppose $A = \begin{array}{c} \\ n_1 \\ n_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \end{array}$ and $B = \begin{array}{c} \\ r_1 \\ r_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} E & F \\ G & H \end{bmatrix} \end{array}$

then the matrices P, Q, R, S and E, F, G, H are called the blocks of the matrices A and B, respectively

Even if A+B is defined

the order of P and E may not be same and hence we may not be able to add A and B in the block form. But if A + B and P + E is defined then

$$A+B = \begin{bmatrix} P+E & Q+F \\ R+G & S+H \end{bmatrix}$$

Similarly if the product AB is defined the product PE need not be defined, therefore we can talk of matrix product AB as block product of matrices if both the products AB and PE are defined and in this case we have

$$AB = \begin{bmatrix} PE+QG & PF+QH \\ RE+SG & RF+SH \end{bmatrix}$$

i.e. Once the partition of A is fixed the partition of B has to be properly choosen for purposes of block addition or multiplication