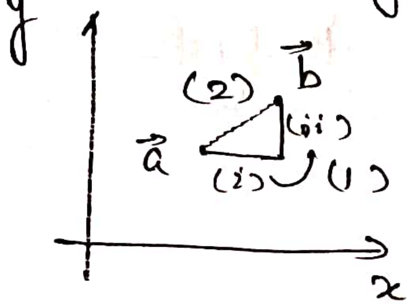
Example :

Calculate the line integral of the function $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point $a = (1,1,0)$ to the point $b = (2,2,0)$ along the paths (1) and (2) as shown in figure below.



What is $\oint \vec{v}\cdot d\vec{l}$ for the loop that goes from $\vec{a}$ ~~an~~ to $\vec{b}$ along (1) and returns to $\vec{a}$ along (2).

Solution: As we choose $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

Along path (1) ; Path (1) consist of two parts

Along horizontal segment $dy = dz = 0$ so

(i) $d\vec{l} = dx\,\hat{x}$, $y = 1$

$$\therefore \vec{v}\cdot d\vec{l} = y^2 dx = dx$$

$$\int \vec{v}\cdot d\vec{l} = \int_1^2 dx = 1$$

On vertical stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\,\hat{y}$ $\quad x = 2$

$$\vec{v}\cdot d\vec{l} = 2x(y+1)\,dy = 4(y+1)\,dy$$

$$\int \vec{v}\cdot d\vec{l} = 4\int_1^2 (y+1)\,dy = 10.$$

By path(1) then

$$\int_a^b \vec{v}\cdot d\vec{l} = 1+10 = 11$$

Meanwhile on path(2) $x=y$; $dx=dy$ and $dz=0$

so $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$$\vec{v}\cdot d\vec{l} = x^2 dx + 2x(x+1)\,\cancel{dy}\,dx$$

$$= (3x^2+2x)\,dx$$

$$\int \vec{v}\cdot d\vec{l} = \int_1^2 (3x^2+2x)\,dx = \cancel{3}\left(x^3+x^2\right)\Big|_1^2 = 10$$

For the loop that goes out(1) and back(2)

then $\oint \vec{v}\cdot d\vec{l} = 11-10 = 1$.