Name of college — S. S. college, Jehanabad.

Deptt — Mathematics

Topic — Fundamental Properties of Definite integrals

class → B. Sc.(Hons)

Time – 10.15 A.M to 11.00 A.M

11.00 A.M to 11.45 A.M

Date — 10-08-2020

Study Material: → By – Dr. Shree Nath Sha

(i) $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

(ii) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(iii) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(iv.) $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,d$$

(vi) show that-

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(v) Put- $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

put $a-x = z$ so that $dx = -dz$

when $x=0$ $z=a$

when $x=a$ $z=0$

$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(z)(-dz)$

$= -\int_a^0 f(z)\,dz$

$= \int_0^a f(z)\,dz = \int_0^a f(x)\,dx.$

Show that $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

Since $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

Put $2a - y = x$ in the 2nd integral

$\Rightarrow -dy = dx$

When $x = a$ $y = a$

$x = 2a$ $y = 0$.

we get $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-y)\,dy = \int_0^a f(2a-y)\,dy$

$= \int_0^a f(2a-x)\,dx$

$= \int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= -\int_0^a f(x)\,dx$ if $f(2a-x) = -f(x)$

$\therefore$ From (1) When $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

When $f(2a-x) = -f(x)$

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 0$$

<u>Show that</u>

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f(x) + f(-x) = 0$$

i.e. $f(x)$ is an odd function

$$= 2\int_0^{a} \quad \text{if } f(x) - f(-x) = 0$$

i.e. $f(x)$ is an even function

$$\text{Now } \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{--- (1)}$$

put $x = -z$ so that $dx = -dz$

When $x = -a$ $z = a$ When $x = 0$ $z = 0$

$$\text{we have } \int_{-a}^{a} f(x)\,dx = \int_{+a}^{0} f(-z)(-dz) = -\int_{a}^{0} f(-z)\,dz$$

$$= \int_0^{a} f(-z)\,dz = \int_0^{a} f(-x)\,dx \quad \text{(2)}$$

$$= \int_0^{a} f(x)\,dx \quad \text{when } f(x) = f(-x)$$

$$= \int_0^{a} f(x)\,dx \quad \text{if } f(x) = -f(x)$$

$\therefore$ from ①

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$= 0$ When $f(x)$ is odd using ②

Also from ①

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$= 2\int_{0}^{a} f(x)\,dx$ When $f(x)$ is an even function

Problem :→ Evaluate

$$\int_{0}^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$$

Sol- $I = \int_{0}^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$

Since $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$

$$\therefore I = \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+[\cos(\pi - x)]^2}\,dx$$

$$= \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+[-\cos x]^2}\,dx$$

$$= \pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - \int_{0}^{\pi} \frac{x \sin x\,dx}{1+\cos^2 x}$$

$$I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I$$

$$2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x},$$

put $\cos x = z \Rightarrow \cos^2 x = z^2$

$\sin x \, dx = -dz$

When $x = \pi$ $z = \cos\pi = -1$

When $x = 0$ $z = \cos 0 = 1$

$$= -\pi \int_1^{-1} \frac{dz}{1+z^2}$$

$$= +\pi \int_{-1}^{1} \frac{dz}{1+z^2} = \pi \left[\tan^{-1} z\right]_0^1$$

$$= \pi \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \pi \left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi^2}{4}$$

2. Evaluate $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}$

$$I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)}{1+(\cos(\pi - x))^2}\,dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$$

$$2I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{\cos^2 x + \sin^2 x + \cos^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{2\cos^2 x + \sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{\cos^2 x \left(2 + \frac{\sin^2 x}{\cos^2 x}\right)}$$

$$= \pi \cdot 2 \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x}\,dx$$

Put $\tan x = z$

$\sec^2 x\,dx = dz$

When $x = 0$ $z = 0$

$x = \pi/2$ $z = \infty$

$$= 2\pi \int_0^{\pi/2} \frac{dz}{z^2 + (\sqrt{2})^2}$$

$$= 2\pi \cdot \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{z}{\sqrt{2}}\right]_0^{\infty}$$

$$2I = \frac{2\pi}{\sqrt{2}} \left[\tan^{-1}\infty - \tan^{-1} 0\right] = \frac{2\pi}{\sqrt{2}} \left[\pi/2 - 0\right]$$

$$\therefore I = \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2}\right) = \frac{\pi^2}{2\sqrt{2}}$$

Problem :→ show

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

Since we have

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$$

$$\therefore I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\sin x + \cos x}$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\, dx.$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)}\, dx$$

$$\frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{1}{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}}\, dx$$

Q-28 $= \frac{2}{\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\sin(x+\pi/4)}$

$\frac{2}{\sqrt{2}} \int_0^{\pi/4} \operatorname{cosec}\left(x+\frac{\pi}{4}\right) dx$

$\log \int \operatorname{cosec} x \, dx$

$\frac{2}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \operatorname{cosec}\left(x+\frac{\pi}{4}\right) dx$

$= \log|\operatorname{cosec} x - \cot x|$

$= \log \frac{1-\cos x}{\sin x}$

$I = \frac{2}{\sqrt{2}} \left[\log \tan\left(\frac{x}{2}+\pi/8\right)\right]_0^{\pi/4}$

$= \log \frac{2\sin^2 x/2}{2\sin x/2 \cos x/2}$

$I = \frac{1}{\sqrt{2}} \left[\log \tan\left(\frac{\pi}{8}+\pi/8\right) - \log \tan \pi/8\right]$

$= \log \tan x/2$

$= \frac{1}{\sqrt{2}} \log \frac{\tan 2\pi/8}{\tan \pi/8}$

$\frac{2\pi}{8} = \pi + \pi/8$

$= \pi/4$

$= \frac{1}{\sqrt{2}} \log \cot \pi/8$

$\frac{\tan \frac{\pi}{4}}{\tan \pi/8}$

$= \frac{1}{\sqrt{2}} \log \tan\frac{\pi}{4} \cdot \cot\frac{\pi}{8}$

$= \frac{1}{\sqrt{2}} \log \cot \pi/8$

$\cot \pi/8 = \sqrt{2}+1$

$= \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$