Name of college -
S. S. College, J. Bad
Deptt - Mathematics

Topic - Integration
Time - 10.15 A.M To 11.00 A.M
11.00 A.M To 11.45 A.M
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Class - B.Sc I (HONS)

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Study Material

If $\frac{d[F(x)+C]}{dx} = F'(x) \Rightarrow \int F'(x)\,dx = F(x) + C$

Some Standard Form :→

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$ Where $n+1 \neq 0$ i-e $n \neq -1$ $\quad \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$\int \frac{1}{x}\,dx = \log x \quad \int e^{ax+b} = \frac{1}{a} e^{ax+b}$

$\int a^x\,dx = \frac{a^x}{\log_e a} = \log_a e \; a^x \quad \int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\log_e a}$

$\int \sin x\,dx = -\cos x \quad \int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a}$

$\int \cos x\,dx = \sin x \quad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}$

$\int \sec^2 x\,dx = \tan x \quad \int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a}$

$\int \operatorname{cosec}^2 x\,dx = -\cot x \quad \int \operatorname{cosec}^2(ax+b)\,dx = \frac{-\cot(ax+b)}{a}$

$\int \sec x \tan x\,dx = \sec x \quad \int \sec(ax+b)\tan(ax+b)\,dx = \frac{\sec(ax+b)}{a}$

$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x \quad \int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = \frac{-\operatorname{cosec}(ax+b)}{a}$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} x/a$

$\int \frac{dx}{1+x^2} = \tan^{-1} x \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1} x/a$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x$ $\quad$ $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1} x/a$

$\int \cosh x\,dx = \sinh x$ $\quad$ $\int \cosh(ax+b)\,dx = \frac{\sinh(ax+b)}{a}$

$\int \sinh x\,dx = \cosh x$ $\quad$ $\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b)$

$\int \operatorname{sech}^2 x\,dx = \tanh x$ $\quad$ $\int \operatorname{sech}^2(ax+b)\,dx = \frac{\tanh(ax+b)}{a}$

$\int \operatorname{cosech}^2 x\,dx = -\coth x$ $\quad$ $\int \operatorname{cosech}^2(ax+b)\,dx = -\frac{\coth(ax+b)}{a}$

$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$ $\quad$ $\int \operatorname{sech}(ax+b)\tanh(ax+b)\,dx = -\frac{\operatorname{sech}(ax+b)}{a}$

$\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x$

$\int \tan x\,dx = \log \sec x$ $\quad$ $\int \cot x\,dx = \log \sin x$

$\int \operatorname{cosec} x\,dx = \log \tan x/2$ $\quad$ $\int \sec x\,dx = \log(\sec x + \tan x)$

$\int \tanh x\,dx = \log \cosh x$ $\quad$ $\int \coth x\,dx = \log \sinh x$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} = \log \frac{x+\sqrt{x^2+a^2}}{a} = \log\left[x+\sqrt{x^2+a^2}\right]$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} x/a = \log\left[x+\sqrt{x^2-a^2}\right]$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$$\int \frac{1}{x^2-a^2}\,dx \quad x>a \quad = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\int \frac{dx}{a^2-x^2} \quad x<a \quad = \frac{1}{2a}\log\frac{a+x}{a-x}$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}x/a$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$$

Before passing on the problem we evaluate some additional standard integrals

(1) $\int \frac{dx}{x^2+a^2}$

put $x = a\tan\theta \Rightarrow \tan\theta = x/a$

$dx = a\sec^2\theta\, d\theta \Rightarrow \theta = \tan^{-1} x/a$

$x^2+a^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}x/a$$

(ii) $\int \frac{dx}{x^2-a^2} \quad x > a$

$= \int \frac{1}{(x-a)(x+a)} dx = \frac{1}{2a} \int \frac{(x+a)-(x-a)}{(x-a)(x+a)} dx$

$= \frac{1}{2a}\int \frac{1}{x-a} dx - \frac{1}{2a}\int \frac{1}{x+a} dx$

$= \frac{1}{2a} \log(x-a) - \frac{1}{2a} \log(x+a)$

$= \frac{1}{2a} [\log(x-a) - \log(x+a)]$

$= \frac{1}{2a} \log \frac{x-a}{x+a}$

or put $x = a\coth\theta \Rightarrow dx = -a\,\mathrm{cosech}^2\theta\, d\theta$

$x^2-a^2 = a^2\mathrm{cosech}^2\theta$

$\Rightarrow = \int \frac{-a\,\mathrm{cosech}^2\theta\, d\theta}{a^2\,\mathrm{cosech}^2\theta} = -\frac{1}{a}\int d\theta = -\frac{1}{a}\theta = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} \quad x < a$

$\int \frac{dx}{a^2-x^2} = \frac{(a+x)+(a-x)}{(a+x)(a-x)} = \frac{1}{2a}\int \left[\frac{a-x+a+x}{(a+x)(a-x)}\right] dx$

$= \frac{1}{2a}\int \frac{1}{a+x} dx + \frac{1}{2a}\int \frac{dx}{a-x}$

$= \frac{1}{2a} \log(a+x) - \frac{1}{2a} \log(a-x)$

$= \frac{1}{2a} \log \frac{a+x}{a-x}$

Q. $\int \frac{dx}{a^2-x^2}$ put $x = a\tanh\theta$ $\quad \tanh\theta = \frac{x}{a}$

$\Rightarrow dx = a\,\mathrm{sech}^2\theta$ $\quad \theta = \tanh^{-1}\frac{x}{a}$

$a^2-x^2 = a^2(1-\tanh^2\theta) = a^2\,\mathrm{sech}^2\theta$

$$= \int \frac{a\,\mathrm{sech}^2\theta}{a^2\,\mathrm{sech}^2\theta}\,d\theta = \frac{1}{a}\theta = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

2. $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

Put $x = a\sinh t$ $\quad \sinh t = x/a$

$dx = a\cosh t\,dt$ $\quad \Rightarrow t = \sinh^{-1}\frac{x}{a}$

$$\int \sqrt{x^2+a^2}\,dx = \int \sqrt{a^2\sinh^2 t + a^2}\;a\cosh t\,dt$$

$$= \int \sqrt{a^2(\sinh^2 t+1)}\cdot a\cosh t\,dt$$

$$= \int a^2\cosh^2 t\,dt$$

$$= \frac{a^2}{2}\int (\cosh 2t + 1)\,dt$$

$$= \frac{a^2}{2}\left[\int \cosh 2t\,dt + \int dt\right]$$

$$= \frac{a^2}{2}\left[\sinh 2t\;\cancel{dt} + t\right]$$

$= \frac{a^2}{2}\frac{\sinh 2t}{2} + \frac{a^2}{2}t.$

$= \frac{a^2}{\cancel{2}}\frac{\cancel{2}\sinh t\cosh t}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$= \frac{a^2}{2}\frac{x}{a}\sqrt{1+x^2/a^2} + \frac{a^2}{2}\sinh^{-1}x/a$

$= \frac{\cancel{a^2}}{2}\frac{x}{\cancel{a^2}}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}x/a$

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$\because \sinh^{-1}\frac{x}{a} = \log\left[\frac{x}{a} + \sqrt{x^2/a^2+1}\right]$

$= \log\frac{x+\sqrt{x^2+a^2}}{a}$

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2+a^2}}{a}\right]$

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right] - \frac{a^2}{2}\log a$

Omitting the constant

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

* $\int\sqrt{x^2-a^2}\,dx$

put $x = a\cosh\theta$ $\quad\theta = \cosh^{-1}x/a$

$dx = a\sinh\theta$ $\quad x^2-a^2 = a^2(\cosh^2\theta-1)$

$= a^2\sinh^2\theta$

$= \int \frac{a\sinh\theta\cdot a\sinh\theta\,d\theta}{}$

$= a^2\int\sinh^2\theta\,d\theta$

$= \frac{a^2}{2}\int (\cosh 2\theta - 1)\, d\theta$

$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right]$

$= \frac{a^2}{2}\left[\frac{\cancel{2}\sinh\theta\cosh\theta}{\cancel{2}} - \theta\right]$

$= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$

* $\int\sqrt{a^2-x^2}\, dx$

put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta$

$= \int a\cos\theta \times a\cos\theta\, d\theta$

$= a^2\int\frac{1+\cos 2\theta}{2}\, d\theta$

$= a^2\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{2}\right]$

$= a^2\left[\frac{1}{2}\sin^{-1}x/a + \frac{2\sin\theta\cos\theta}{2}\right]$

$= \frac{a^2}{2}\sin^{-1}x/a + \frac{a^2}{2}\frac{x}{a}\sqrt{1-x^2/a^2}$

$= \frac{a^2}{2}\sin^{-1}x/a + \frac{x\sqrt{a^2-x^2}}{2}$

$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}x/a$

* $\int \frac{dx}{\sqrt{x^2+a^2}} = \log[x+\sqrt{x^2+a^2}]$

Put $x+\sqrt{x^2+a^2} = z$

$\left[1+\frac{1}{2\sqrt{x^2+a^2}} \times 2x\right] dx = dz$

$= \left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right] dx = dz$

$\frac{z}{\sqrt{x^2+a^2}} dx = dz$

$\frac{dx}{\sqrt{x^2+a^2}} = \frac{dz}{z}$

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{dz}{z} = \log z = \log[x+\sqrt{x^2+a^2}]$

$= \sinh^{-1}\frac{x}{a}$

* $\int \frac{dx}{\sqrt{x^2-a^2}} = \log[x+\sqrt{x^2-a^2}] = \cosh^{-1} x/a. \quad x>a.$

Put $z = x+\sqrt{x^2-a^2}$

$dz = \left[1+\frac{1}{2\sqrt{x^2-a^2}} \times 2x\right] dx$

$= \frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}} dx$

$\therefore = \frac{z}{\sqrt{x^2-a^2}} dx$

$\therefore \frac{dz}{z} = \frac{dx}{\sqrt{x^2-a^2}}$

$= \int \frac{dz}{z} = \log z = \log[x+\sqrt{x^2-a^2}]$