Spherical Coordinates (r, θ, ϕ)

The spherical coordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry.

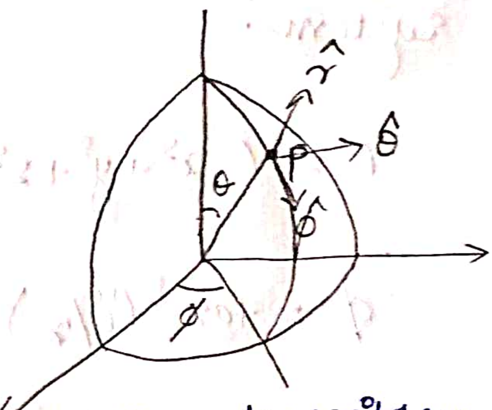
A point P can be represented as (r, θ, ϕ) here we notice that r is defined as the distance from the origin to point P or the radius of sphere centered at the origin and passing through P;

θ is called as colatitude, is the angle between z-axis and the position vector of P;

ϕ is called as azimuthal angle, measured from x axis in xy plane.

According to these definitions, the ranges of the variables are

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$



A vector $\vec{A}$ in spherical coordinates may be written as

$$(A_r, A_\theta, A_\phi) \quad \text{or} \quad (A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi})$$

where $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are unit vector along r, θ and ϕ directions.

The unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are mutually orthogonal, $\hat{r}$ being directed along the radius

$\hat{\theta}$ in the direction of increasing θ

$\hat{\phi}$ in the direction of increasing ϕ

Thus,

$$\left.\begin{aligned} \hat{r}\cdot\hat{r} &= \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1 \\ \hat{r}\cdot\hat{\theta} &= \hat{\theta}\cdot\hat{\phi} = \hat{r}\cdot\hat{\phi} = 0 \end{aligned}\right] \text{Scalar product}$$

$$\left.\begin{aligned} \hat{r}\times\hat{\theta} &= \hat{\phi} \\ \hat{\theta}\times\hat{\phi} &= \hat{r} \\ \hat{\phi}\times\hat{r} &= \hat{\theta} \end{aligned}\right] \text{cross product}$$

The space variables (x, y, z) in cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system.

$$r = (x^2+y^2+z^2)^{1/2} \quad ; \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \tan^{-1}(y/x)$$

or

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Relation between $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{r}, \hat{\theta}, \hat{\phi})$

$$\hat{x} = \sin\theta \cos\phi \, \hat{r} + \cos\theta \cos\phi \, \hat{\theta} - \sin\phi \, \hat{\phi}$$

$$\hat{y} = \sin\theta \sin\phi \, \hat{r} + \cos\theta \sin\phi \, \hat{\theta} + \cos\phi \, \hat{\phi}$$

$$\hat{z} = \cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}$$

or

$$\hat{r} = \sin\theta \cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

$$\hat{\theta} = \cos\theta \cos\phi \, \hat{x} + \cos\theta \sin\phi \, \hat{y} - \sin\theta \, \hat{z}$$

$$\hat{\phi} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y}$$

Transformation of vector

$$(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} \cancel{\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}} = \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Transformation

$$(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$