The curl

From the definition of $\vec{\nabla}$ we construct the curl:
where $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{v}$ is, like any cross product, a vector.

You cannot have the ~~curl~~ curl of a scalar; that is meaningless.

<u>Example</u>: Suppose the function $v_a = -y\hat{x} + x\hat{y}$ and $v_b = x\hat{y}$. Calculate their curls.

solution

$$\vec{\nabla}\times\vec{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right) + \hat{y}\left(\frac{\partial}{\partial z}(-y) - \frac{\partial 0}{\partial x}\right) + \hat{z}\left(\frac{\partial x}{\partial x} - \frac{\partial}{\partial y}(-y)\right)$$

$$= \hat{z}\,(1+1) = 2\hat{z}$$

and $\vec{v}_b = x\hat{y}$

$$\vec{\nabla}\times\vec{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x\right) + \hat{y}\left(\frac{\partial}{\partial z}0 - \frac{\partial}{\partial x}0\right) + \hat{z}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}0\right)$$

$$= \hat{z}$$

The Fundamental Theorem for Curls

The fundamental theorem for curls, which goes by the special name of ~~Stokes's the~~ Stokes' theorem states that:

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$$

As always, the integral of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary.