Name of College - S.S. College, J-Bad
Deptt - Mathematics
TOPIC - Reduction formula.
Class - B.Sc I (HONS)
Time - 10:15 A.M to 11:00 A.M
11:00 A.M to 11:45 A.M
Date - 18-09-2020

By - Dr. Shree Nath Sharma.

Obtain a reduction formula for
(i) $\int \sin^m x \cos^n x\, dx$ by connecting it with
$\int \sin^{m-2} x \cos^n x\, dx$
In other words if $I_{m,n} = \int \sin^m x \cos^n x\, dx$, Establish $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$

(ii) obtain a reduction formula for
$\int \sin^m x \cos^n x\, dx$ by connecting it with
$\int \sin^m x \cos^{n-2} x\, dx$

In other words, if
$I_{m,n} = \int \sin^m x \cos^n x\, dx$, Establish

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

i-e $(m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}$

(i) In the two integrals, the smaller index of $\sin x$ is $m-2$ and that of $\cos x$ is n

$\therefore \lambda = m-2 \quad \mu = n$

So here $P = \sin^{\lambda+1} x \cos^{\mu+1} x = \sin^{m-1} x \cos^{n+1} x$

$$\therefore \frac{dP}{dx} = (m-1)\sin^{m-2} x \cos x \cdot \cos^{n+1} x + (n+1)\sin^{m-1} x \cos^{n} x(-\sin x)$$

$$= (m-1)\sin^{m-2} x \cos^{n} x \cos^2 x - (n+1)\sin^{m} x \cos^{n} x$$

$$= (m-1)\sin^{m-2} x \cos^{n} x(1-\sin^2 x) - (n+1)\sin^{m} x \cos^{n} x$$

$$= (m-1)\sin^{m-2} x \cos^{n} x - (m-1)\sin^{m} x \cos^{n} x - (n+1)\sin^{m} x \cos^{n} x$$

$$= (m-1)\sin^{m-2} x \cos^{n} x - (m-1)\sin^{m} x \cos^{n} x - (n+1)\sin^{m} x \cos^{n} x$$

$$= (m-1)\sin^{m-2} x \cos^{n} x - [(m-1)+(n+1)]\sin^{m} x \cos^{n} x$$

$$= (m-1)\sin^{m-2} x \cos^{n} x - (m+n)\sin^{m} x \cos^{n} x$$

Integrating, we get-

$$P = (m-1)\int \sin^{m-2} x \cos^{n} x \, dx - (m+n)\int \sin^{m} x \cos^{n} x \, dx$$

$$\therefore \int \sin^{m} x \cos^{n} x \, dx = -\frac{P}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^{n} x \, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^{n} x \, dx$$

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

or $(m+n) I_{m,n} = (m-1) I_{m-2,n} - \sin^{m-1} x \cos^{n+1} x$

We are to find the reduction formula (2) for $\int \sin^m x \cos^n x \, dx$ by connecting it with $\int \sin^m x \cos^{n-2} x \, dx$

Here the smaller indices are m and $n-2$, so we take

$$P = \sin^{\lambda+1} x \cos^{\mu+1} x = \sin^{m+1} x \cos^{(n-2)+1} x = \sin^{m+1} x \cos^{n-1} x$$

$$\therefore \frac{dP}{dx} = (m+1) \sin^m x \cos x \cos^{n-1} x - (n-1) \sin^{m+1} x \cos^{n-2} x \sin x$$

$$= (m+1) \sin^m x \cos^n x - (n-1) \sin^{m+2} x \cos^{n-2} x$$

$$= (m+1) \sin^m x \cos^n x - (n-1) \sin^m x \cos^{n-2} x (1-\cos^2 x)$$

$$= (m+1) \sin^m x \cos^n x - (n-1) \sin^m x \cos^{n-2} x + (n-1) \sin^m x \cos^n x$$

$$= \{m+1+n-1\} \sin^m x \cos^n x - (n-1) \sin^m x \cos^{n-2} x$$

Integrating, we get

$$P = (m+n) \int \sin^m x \cos^n x \, dx - (n-1) \int \sin^m x \cos^{n-2} x \, dx$$

$$\therefore \int \sin^m x \cos^n x \, dx = \frac{P}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$$

i.e. $$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\Rightarrow (m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}$$

Obtain a Reduction formula for (4)

$\int \cos^m x \sin nx \, dx$

Let $I_{m,n} = \int \cos^m x \sin nx \, dx$

Integrating by parts

$$= -\frac{\cos^m x \cos nx}{n} - \int \left[\int \sin nx \, dx \cdot \frac{d}{dx}\cos^m x\right] dx$$

$$= -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} \int \cos nx \cos^{m-1} x \, (-\sin x) \, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \cos nx \sin x \, dx$$

But, $\sin(n-1)x = \sin(nx - x)$

$$= \sin nx \cos x - \cos nx \sin x$$

$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

$$\therefore I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \left\{ \sin nx \cos x - \sin(n-1)x \right\} dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^m x \sin nx \, dx + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x \, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{m+n} I_{m-1,n-1}$$

In same way
reduction formula for (i) $\int \cos^m x \cos nx \, dx$

(ii) $\int \sin^m x \sin nx \, dx$

may be obtained.

Beta and Gamma Function:

The Beta function or the first Eulerian integral is defined by the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$,

Where m and n are supposed positive in all cases. It is denoted by $B(m,n)$

Gamma Function :-) The definite integral

$\int_0^\infty e^{-x} x^{n-1} dx$ is known as the gamma function or the second Eulerian integral denoted by Γn and read as gamma n.

Thus $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Where n is +ve number, integer or fraction.

Properties of gamma function :— $\Gamma 1 = 1$ $\Gamma 0 = \infty$ $\Gamma -n = \infty$, $\Gamma n+1 = n \Gamma n$ $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Gamma n = \lfloor n-1$

i.e $\Gamma n = (n-1)(n-2) \ldots 2 \cdot 1 \; \Gamma 1$, n being be +ve integer.

$\Gamma 7/2 = \Gamma(5/2+1) = \frac{5}{2} \Gamma 5/2$

$= \frac{5}{2} \Gamma 3/2+1$

$= \frac{5}{2} \times \frac{3}{2} \Gamma 3/2$

$= \frac{5}{2} \times \frac{3}{2} \Gamma \frac{1}{2}+1$

$= \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \Gamma 1/2$

$= \frac{15}{8} \sqrt{\pi}$.

Similarly for gamma of half of any odd No.

Further $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$.