Gauss' Law in differential form

When a charge is distributed over a volume such that ρ is the density of charge. Then the charge enclosed by surface enclosing the volume is given by

$$q = \int_V \rho \, dv$$

using Gauss Law

$$\oint \vec{E} \cdot d\vec{s} = \frac{\int_V \rho \, dv}{\epsilon_0}$$

Now applying divergence theorem

$$\oint_S \vec{E} \cdot d\vec{s} = \int_V \vec{\nabla} \cdot \vec{E} \, dv$$

so, $$\int_V \vec{\nabla} \cdot \vec{E} \, dv = \frac{\int_V \rho \, dv}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

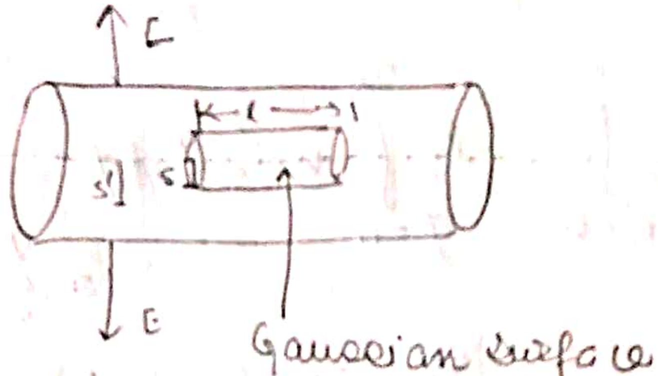
For any volume integrands it must be equal

∴ we say divergence of electric field is proportional to the charge at that point.

Physically we can interpret it that, flux of $\vec{E}$ directly depends on the magnitude of charge or charge density.

Q. A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks'$, for some constant k. Find the electric field inside this cylinder.

<u>Solution</u>



Drawing a Gaussian cylinder of length l and radius s. For this surface, Gauss' law states:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{inside}$$

$$Q_{inside} = \int \rho \, d\tau$$

where ρ is charge density
$d\tau$ is volume element

$$d\tau = s' ds' \, d\phi \, dz$$

the integrand will have value s' from 0 to s
ϕ from 0 to 2π
z from 0 to l

$$Q_{inside} = \iiint k s' \, s' \, ds' \, d\phi \, dz$$

$$= \int_{s'=0}^{s} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} k s'^2 \, ds' \, d\phi \, dz$$

$$= \int_{s'=0}^{s} \int_{\phi=0}^{2\pi} k s'^2 ds' \, d\phi \int_{z=0}^{l} dz$$

$$= \int_{s'=0}^{s} \int_{\phi=0}^{2\pi} k s'^2 ds' \, d\phi \, (l-0)$$

$$= l \int_{s'=0}^{s} k s'^2 ds' \int_{\phi=0}^{2\pi} d\phi$$

$$= 2\pi l k \int_{s'=0}^{s} s'^2 ds' = 2\pi l k \left| \frac{s'^3}{3} \right|_0^s$$

$$= \frac{2\pi l k s^3}{3}$$

Now symmetry dictates that $\vec{E}$ must point radially outward, so for the curved portion of the Gaussian cylinder we have

$$\int \vec{E} \cdot d\vec{a} = |\vec{E}| \, 2\pi s l = \frac{2}{3\epsilon_0} \pi l k s^3$$

$$|\vec{E}| = \frac{1}{3\epsilon_0} k s^2$$