**College: S. S. College. Jehanabad (Magadh University)**

**Subject : Quantum Mechanics**

**Class : B.Sc( H) Physics Part III**

**Topic: Heisenberg Uncertainty Proof.**

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## Ehrenfest's Theorem

According to Ehrenfest's theorem, the expectation values of the position and momentum vectors for a wave packets are formally identical to newton's equations of classical mechanics:

$$\frac{d}{dt}\langle r\rangle = \frac{\langle p\rangle}{m}$$

and

$$\frac{d}{dt}\langle p\rangle = -\langle \nabla V\rangle$$

## Heisenberg Uncertainity Relation Proof

It is simple to define uncertainity as the root mean square deviation from the mean value:

$$\Delta x = \langle (x-\langle x\rangle)^2\rangle^{1/2}$$

$$\Delta p = \langle (p-\langle p\rangle)^2\rangle^{1/2}$$

Let

$$A = x-\langle x\rangle$$

$$B = p-\langle p\rangle = i\hbar\left[\frac{d}{dx}-\left\langle\frac{d}{dx}\right\rangle\right]$$

Then

$$(\Delta x)^2(\Delta p)^2 = \int_{-\infty}^{\infty}\psi^* A^2\psi\, dx \int_{-\infty}^{\infty}\psi^* B^2\psi\, dx$$

$$= \int_{-\infty}^{\infty}(A^*\psi^*)(A\psi)\, dx \int_{-\infty}^{\infty}(B^*\psi^*)(B\psi)\, dx$$

from the schwarz inequality

$$\int |f|^2 dx \int |g|^2 dx \geq \left|\int f^* g\, dx\right|^2$$

$$(\Delta x)^2(\Delta p)^2 \geq \left|\int (A^*\psi^*)(B\psi)\, dx\right|^2 = \left|\int \psi^* AB\psi\, dx\right|^2$$

$\longrightarrow$ (i)

$$\left[\int \psi^* AB\psi\, dx\right]^* = \int \psi A^* B^* \psi^* dx$$

$$= \int B^* \psi^* A\psi\, dx$$

$$= \int \psi^* BA\, \psi\, dx$$

$$(AB - BA)\psi = -i\hbar\left[x\frac{d\psi}{dx} - \frac{d}{dx}(x\psi)\right]$$

$$= -i\hbar\left[x\frac{d\psi}{dx} - x\frac{d\psi}{dx} - \psi\right]$$

$$= +i\hbar\psi$$

Therefore

$$\int \psi^* (AB - BA)\psi\, dx = i\hbar \int \psi^* \psi\, dx$$

$$= i\hbar \quad \text{— (2)}$$

from (1), R.H.S side can be written as

$$\left|\int \psi^* AB\psi\, dx\right|^2 = \left|\int \psi^* \left[\tfrac{1}{2}(AB - BA) + \tfrac{1}{2}(AB + BA)\right]\psi\, dx\right|^2 \quad \text{— (3)}$$

using (2) & (3), we get

$$(\Delta x)^2 (\Delta p)^2 \geq \hbar^2/4$$

$$\Delta x \Delta p \geq \hbar/2$$