Name of College - S. S. College, J. Bad. 1

Deptt - Mathematics

Topic :- Definite integration as limit of a sum.

Class - B. Sc I (Hons)

Time - 10.15 A.M to 11 A.M

11.00 A.M to 11.45 A.M

Date - 14-09-2020

By Dr. Shree Nath Sharma

If $f(x)$ is a single-valued and continuous function in the interval (a,b) where $b>a$ and if the interval (a,b) be divided into n equal sub-intervals each of length h by points

$a+h,\ a+2h,\ a+3h\ \cdots\ +a+(n-1)h.$

Where $nh=b-a$ then

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$$

$$\text{or } \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

Integration by summation is also called integration from first principle or integration by definition

* Evaluate the following by summation

$$\int_a^b x\,dx$$

Here $f(x) = x$

$f(a+rh) = a+rh$

Now $\int_a^b x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} a+rh$

$= \lim_{h\to 0} h\left[a\sum_{r=1}^{n} 1 + h\sum_{r=1}^{n} r\right]$ $\qquad \sum_{r=1}^{n} 1 = n$

$= \lim_{h\to 0} h\left[an + h\frac{n(n+1)}{2}\right]$ $\qquad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$= \lim_{h\to 0}\left[anh + \frac{h^2 n(n+1)}{2}\right]$

$= a\lim_{h\to 0} nh + \lim_{h\to 0}\frac{nh(nh+h)}{2}$

$= a(b-a) + \frac{(b-a)(b-a+0)}{2}$

$= a(b-a) + \frac{(b-a)^2}{2}$

$= \frac{2a(b-a)+(b-a)^2}{2}$

$= \frac{(b-a)[2a+b-a]}{2}$

$\frac{(b-a)(b+a)}{2} = \frac{b^2-a^2}{2}$

* $\int_a^b x^2 dx$

Here $f(x) = x^2$.

$nh = b-a$

$f(a+rh) = (a+rh)^2$

$= a^2 + 2arh + r^2h^2$

Now from the definition

$$\int_a^b x^2 dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h\to 0} h \left[\sum_{r=1}^{n} a^2 + 2arh + r^2h^2\right]$$

$$= \lim_{h\to 0} a^2 h \sum_{r=1}^{n} 1 + 2ah^2 \sum_{r=1}^{n} r + h^3 \sum_{r=1}^{n} r^2$$

$$= a^2 \lim_{h\to 0} nh + \lim_{h\to 0} 2ah^2 \frac{n(n+1)}{2} + \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= a^2(b-a) + 2a \lim_{h\to 0} \frac{nh(nh+h)}{2} + \frac{1}{6} \lim_{h\to 0} nh(nh+h)(2nh+h)$$

$$= a^2(b-a) + \frac{2a(b-a)(b-a+0)}{2} + \frac{1}{6}(b-a)[b-a+0][2(b-a)+0]$$

$$= a^2(b-a) + \frac{2a(b-a)^2}{2} + \frac{1\times 2}{6}(b-a)^3$$

$$= (b-a)\left[a^2 + a(b-a) + \frac{(b-a)^2}{3}\right]$$

$$= (b-a)\frac{\left[3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right]}{3}$$

$$= (b-a)\left(\frac{a^2+ab+b^2}{3}\right)$$

$$= \frac{1}{3}(b^3 - a^3)$$

(3) $\int_0^1 x^3\,dx$

Here $f(x) = x^3$ $a=0$ $b=1$

$nh = b-a = 1-0 = 1$

$f(a+rh) = f(0+rh) = r^3h^3$

$\therefore \sum f(a+rh) = h^3 \sum_{r=1}^{n} r^3$

$\Rightarrow f(rh)$

$$= \frac{h^3\; n^2(n+1)^2}{4}$$

$\cancel{2\;n^2h^2 (nh+}$

Now $\int_0^1 x^3\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$

$$= \lim_{h\to 0} h.\, h^3 \frac{n^2(n+1)^2}{4}$$

(2n+ 1)

(2nh +1)

+h)

$$\int_0^1 x^2\,dx = \lim_{h\to 0} \frac{n^2h^2\left[nh+h\right]^2}{4}$$

$$= \lim_{h\to 0} \frac{1\cdot(1+h)^2}{4} = \frac{1}{4}$$

*. $\int^b dx$

$\int_a^b f(x)\,dx = \int_a^b \frac{1}{x^2}\,dx$ (6)

Sub divide the interval so that the abscissae of the points of sub-interval are in G.P and we have

$x_0 = a \qquad x_2 = ar^2$

$x_1 = ar \qquad x_3 = ar^3$

$= \; = \; = \; =$

$x_n = ar^n = b$

$$\int_a^b \frac{1}{x^2}\,dx = \lim_{r\to 1} a(r-1)\left[\frac{1}{a^2} + r\cdot\frac{1}{a^2r^2} + r^2\frac{1}{a^2r^4} + \cdots\right]$$

$$= \lim_{r\to 1}\frac{a(r-1)}{a^2}\left[1 + \frac{1}{r} + \frac{1}{r^2} + \cdots\right]$$

$$= \lim_{r\to 1}\frac{a(r-1)}{a^2}\left[\frac{\left(\frac{1}{r}\right)^n - 1}{\frac{1}{r} - 1}\right]$$

$$= \frac{1}{a}\lim_{r\to 1}(r-1)\left[\frac{1-r^n}{(1-r)r^{n-1}}\right]$$

$$= \frac{1}{a}\lim_{r\to 1}\frac{1-\frac{b}{a}\cdot r}{b/a}$$

$$= \frac{1}{a}\,\frac{(a-b)}{b} = -\frac{1}{a}$$

$$= \frac{1}{a}\,\frac{a-b}{b} \quad -\frac{1}{a} - 1/b$$

Q. $\int_a^b \sin x \, dx$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum f(a+rh) \qquad nh = b-a.$$

$$\therefore \int_a^b \sin x \, dx = \lim_{h\to 0} h \left[\sin a + \sin(a+h) + \cdots \text{ upto } n \right]$$

$$= \lim_{h\to 0} h \frac{\sin \frac{nh}{2}}{\sin h/2} \sin\left(a + \frac{n-1}{2} h\right)$$

$$= \lim_{h\to 0} h \cdot \frac{\sin \frac{(b-a)}{2}}{\frac{\sin h/2}{h/2} \times \frac{h}{2}} \sin\left[a + \frac{nh-h}{2}\right]$$

$$= \lim_{h\to 0} \sin\frac{(b-a)}{2} \lim_{h\to 0} \sin\left[\frac{2a+nh-h}{2}\right]$$

$$= \sin\frac{(b-a)}{2} \cdot \times 2 \times \sin\frac{(a+b)}{2}$$

$$= \cos a - \cos b.$$