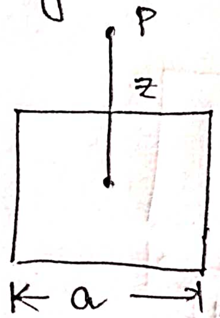
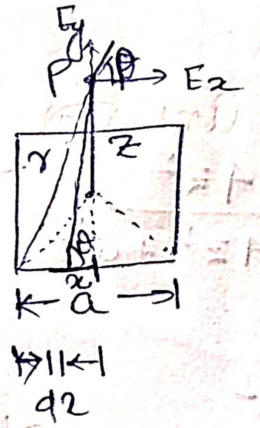
8. Find the electric field a distance z above the center of a square loop (side a) carrying uniform line charge λ.



Answer:



For one side

$$r^2 = z^2 + x^2$$

$$dq = \lambda dx$$

$$\int dE_y = \int_0^{a/2} \frac{2\lambda dx}{4\pi\epsilon_0 r^2} \sin\theta$$

$$\sin\theta = \frac{z}{r}$$

$$\int dE_y = \frac{2\lambda}{4\pi\epsilon_0} \int_0^{a/2} \frac{dx}{r^2} \frac{z}{r}$$

$$\int dE_y = \frac{2\lambda}{4\pi\epsilon_0} \int_0^{a/2} \frac{z}{r^3} dx$$

$$\int dE_y = \frac{2\lambda}{4\pi\epsilon_0} \int_0^{a/2} \frac{z}{(z^2+x^2)^{3/2}} dx$$

let $x = z\tan\theta$

$$(z^2+x^2)^{3/2} = z^3\sec^3\theta$$

$$dx = z\sec^2\theta \, d\theta$$

$$\int dE_y = \frac{2\lambda}{4\pi\epsilon_0} \int_0^{\tan^{-1} a/2z} \frac{z\sec^2\theta \, d\theta}{z^3 \sec^3\theta}$$

$$E_y = \frac{2\lambda}{4\pi\epsilon_0} \frac{1}{z} \int_0^{\tan^{-1}\frac{a}{2z}} \cos\theta \, d\theta$$

$$E_y = \frac{2\lambda}{4\pi\epsilon_0} \frac{1}{z} \left|\sin\theta\right|_0^{\tan^{-1} a/2z}$$

$$E_y = \frac{2\lambda}{4\pi\epsilon_0} \frac{1}{z} \left[\sin(\tan^{-1} a/2z) - \sin 0\right]$$

Let $z = \tan^{-1} a/2z$

$\tan z = a/2z$

$\Rightarrow \quad 1 + \tan^2 z = 1 + \frac{a^2}{4z^2}$

$\Rightarrow \quad \sec^2 z = \frac{4z^2 + a^2}{4z^2}$

$\Rightarrow \quad \cos^2 z = \frac{4z^2}{4z^2 + a^2}$

$\Rightarrow \quad 1 - \cos^2 z = 1 - \frac{4z^2}{4z^2 + a^2}$

$\Rightarrow \quad \sin^2 z = \frac{a^2}{4z^2 + a^2}$

$\Rightarrow \quad \sin z = \frac{a}{\sqrt{4z^2 + a^2}}$

$\therefore \quad E_y = \frac{2\lambda}{4\pi\epsilon_0} \frac{1}{z} \frac{a}{\sqrt{4z^2 + a^2}}$

Four all 4 sides Net Electric field

$$E_{y\,Net} = 4 \times \frac{2\lambda}{4\pi\epsilon_0} \frac{a}{z\sqrt{4z^2 + a^2}}$$

$$= \frac{2\lambda a}{4\pi\epsilon_0 z \sqrt{4z^2 + a^2}}$$