Name of college - S. S. College, J. Bad
Deptt - Mathematics
Topic - Definite integration
class - B.Sc I (Hons)
Time - 10.15 A.M to 11.00 A.M
11.00 A.M to 11.45 A.M
Date - 07-09-2020

1. Fundamental theorem of Integral calculus

If $\int f(x)\,dx = \phi(x) \;\forall\; x \in [a,b]$

$$\Rightarrow \int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$$

Remarks — No constant is added in a definite integral

2. Improper integrals :→

In definite integrals it is supposed that the range of integration is finite and the integrand is continuous in the range.

If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range, the integral is called improper integral or infinite integral

Problem :→ Evaluate the following

1. $$\int_0^{1/2} \frac{dx}{(x-1)\sqrt{x-x^2}}$$

Solution :→ put $x-1 = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\,dz$

When $x = 0 \Rightarrow z = -1$

When $x = \frac{1}{2}$ $\frac{1}{z} = -\frac{1}{2} \Rightarrow z = -2$

therefore, integral $= \int_{-1}^{-2} \frac{\left(-\frac{1}{z^2}\right) dz}{\frac{1}{z}\sqrt{\frac{1}{z}-\frac{1}{z^2}}}$

$$= -\int_{-1}^{-2} \frac{\frac{1}{z^2}\, dz}{\frac{1}{z}\sqrt{\left[\frac{1}{z}+1\right]-\left[\frac{1}{z^2}+1\right]}}$$

$$= -\int_{-1}^{-2} \frac{\frac{1}{z^2}\, dz}{\frac{1}{z}\sqrt{\frac{z+1}{z}\left(1-\frac{1+z}{z}\right)}}$$

$$= -\int_{-1}^{-2} \frac{\frac{1}{z^2}\, dz}{\frac{1}{z}\sqrt{\frac{1+z}{z}\left(-\frac{1}{z}\right)}}$$

$$= -\int_{-1}^{-2} \frac{dz}{(-z-1)^{1/2}}$$

$$= -\int_{-1}^{-2} (-z-1)^{-1/2}\, dz$$

use.

$$\int [ax+b]^n dx = \frac{[ax+b]^{n+1}}{a(n+1)}$$

$$= -\left[\frac{(-z-1)^{1/2}}{(-1)\,1/2}\right]_{-1}^{-2}$$

$$= +2\left[(+2-1)^{1/2} - (1-1)^{1/2}\right]$$

$$= +2\left[\sqrt{1} - 0\right] = 2.$$

2 *. Evaluate $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put. $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2} dz$

When $x=0$ $z = \infty$

When $x = \frac{1}{\sqrt{3}}$ $\Rightarrow z = \sqrt{3}$

therefore the integral $= \int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

$$= \int_{\infty}^{\sqrt{3}} \frac{-\frac{1}{z^2}\,dz}{\left(1+\frac{1}{z^2}\right)\sqrt{1-\frac{1}{z^2}}}$$

$$= \int_{\infty}^{\sqrt{3}} \frac{-\frac{1}{z^2}\,dz}{\frac{z^2+1}{z^2}\sqrt{\frac{z^2-1}{z}}}$$

$$= \int_{\infty}^{\sqrt{3}} \frac{-z\,dz}{(z^2+1)\sqrt{z^2-1}}$$

put $z^2-1 = t^2$

$2z\,dz = 2t\,dt$

When $z=\infty$ $t=\infty$

$z=\sqrt{3}$ $t=\sqrt{2}$ and

the Given integral

$$= -\int_{\infty}^{\sqrt{2}} \frac{\cancel{t}\,dt}{(t^2+2)\cancel{t}}$$

$$= -\int_{\infty}^{\sqrt{2}} \frac{dt}{t^2+2}$$

using $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$= -\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_{\infty}^{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}\left[\tan^{-1}1 - \tan^{-1}\infty\right]$$

$$= -\frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \pi/2\right] = -\frac{1}{\sqrt{2}}\pi\left[\frac{1-2}{4}\right] = \frac{\pi}{4\sqrt{2}}.$$

Problem :→ Evaluate

$$\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$$

Put $x = \tan\theta$. $\theta = \tan^{-1}x$

$dx = \sec^2\theta\, d\theta$ when $x=1$

$\theta = \tan^{-1}1 = \pi/4$

$1+x^2 = 1+\tan^2\theta = \sec^2\theta$ when $x=0$ $\theta = \tan^{-1}0 = 0$

$[1+x^2]^{3/2} = [\sec^2\theta]^{3/2} = \sec^3\theta.$

$$\therefore I = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_0^{\pi/4} \frac{d\theta}{\sec\theta}$$

$$= \int_0^{\pi/4} \cos\theta\, d\theta = [\sin\theta]_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

* Evaluate

$$\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}$$

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta.$

$dx = [\alpha \cdot 2\cos\theta\sin\theta + \beta \times 2\sin\theta\cos\theta]\, d\theta$

$= 2\sin\theta\cos\theta(\beta-\alpha)$ and $\beta - x$

$x - \alpha = \cancel{(\alpha\cos^2\theta - \alpha)(\beta}$

$\alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

$= \beta\sin^2\theta + (1-\cos^2\theta)\alpha$

$= \beta\sin^2\theta - \alpha\sin^2\theta$

$= (\beta-\alpha)\sin^2\theta$

$= \beta - (\alpha\cos^2\theta + \beta\sin^2\theta)$

$= \beta(1-\sin^2\theta) - \alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta.$

Now When $x = \alpha$ $\quad x - \alpha = 0$

$\Rightarrow \sin\theta = 0$ $\quad$ Since $\beta \neq \alpha$

$\therefore \theta = 0$

When $x = \beta$ .. $\quad \beta - x = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$

$$\therefore \text{Integral} = \int_0^{\pi/2} \sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta} \times 2(\beta-\alpha) \sin\theta\cos\theta \, d\theta$$

$$= (\beta-\alpha)^2 \int_0^{\pi/2} 2\sin\theta \cdot \cos\theta \cdot \sin\theta \cdot \cos\theta \, d\theta$$

$$= (\beta-\alpha)^2 \int_0^{\pi/2} \frac{\sin 2\theta \times \sin 2\theta}{2} d\theta \qquad 2\sin\theta\cos\theta = \sin 2\theta$$

$$= \frac{(\beta-\alpha)^2}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta \qquad \sin^2 K\theta = \frac{1-\cos 2K\theta}{2}$$

$$= \frac{(\beta-\alpha)^2}{2} \int_0^{\pi/2} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{(\beta-\alpha)^2}{4} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{(\beta-\alpha)^2}{4} \left[\left(\pi/2 - \frac{\sin 2\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right]$$

$$= \frac{(\beta-\alpha)^2}{4} \left[\left(\frac{\pi}{2} - 0\right) - 0\right]$$

$$= \frac{(\beta-\alpha)^2}{4} \times \frac{\pi}{2} = \frac{(\beta-\alpha)^2 \pi}{8}$$

Problem: → $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx$

Put $x^2 = z$

then $\frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{z}{(z+a^2)(z+b^2)(z+c^2)}$

$$= \frac{A}{(z+a^2)} + \frac{B}{z+b^2} + \frac{C}{z+c^2}$$

$\therefore\ z = A(z+b^2)(z+c^2) + B(z+a^2)(z+c^2) + C(z+a^2)(z+b^2)$

Putting $z = -a^2,\ -b^2,\ -c^2$ successively, we get

$A = -\frac{a^2}{(b^2-a^2)(c^2-a^2)}$, $B = \frac{-b^2}{(a^2-b^2)(c^2-b^2)}$, $C = \frac{-c^2}{(a^2-c^2)(b^2-c^2)}$

$$\therefore \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$$

$$= \int_0^\infty \frac{A\,dx}{x^2+a^2} + \int_0^\infty \frac{B\,dx}{x^2+b^2} + \int_0^\infty \frac{C\,dx}{x^2+c^2}$$

$$= \left[A\cdot\frac{1}{a}\tan^{-1}x/a + B\cdot\frac{1}{b}\tan^{-1}\frac{x}{b} + C\frac{1}{c}\cdot\tan^{-1}\frac{x}{c}\right]$$

$$= \frac{A}{a}\frac{\pi}{2} + B\frac{\pi}{2} + \frac{C}{c}\frac{\pi}{2}$$

$$= \pi/2\left(\frac{A}{a} + \frac{B}{b} + \frac{C}{c}\right)$$

$$= \pi/2\left[-\frac{a}{(b^2-a^2)(c^2-a^2)} - \frac{b}{(a^2-b^2)(c^2-b^2)} - \frac{c}{(a^2-c^2)(b^2-c^2)}\right]$$

$$= \pi/2\left[\frac{a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}\right]$$

$= \frac{\pi}{4}$ [illegible]

$= \frac{\pi}{4}$ [illegible]

Problem : Evaluate

$$\int_0^{\pi/2} \frac{\cos x \, dx}{(2+\sin x)(1+\sin x)}$$

put $\sin x = z \Rightarrow \cos x \, dx = dz$

When $x = 0 \Rightarrow z = 0$ When $x = \pi/2 \Rightarrow z = 1$

$$\therefore \int_0^{\pi/2} \frac{\cos x \, dx}{(2+\sin x)(1+\sin x)} = \int_0^1 \frac{dz}{(z+1)(z+2)}$$

$$= \int_0^1 \left[\frac{1}{1+z} - \frac{1}{2+z} \right] dz$$

$$= \int_0^1 \log(1+z) - \log(2+z) \Big]_0^1$$

$$= (\log 2 - \log 1) - (\log 3 - \log 2)$$

$$= 2 \log 2 - \log 3$$

$$= \log 4 - \log 3$$

$= \log \frac{4}{3}$. This is the required solution