Name of College - S.S. College, J-Bad
Dept - Mathematics
Topic - Hermitian and Skew Hermitian Matrix
Class - B.Sc I (Sub + Hons)
Time - 1:00 P.M to 1:45 P.M
Date - 16-09-2020
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Show that, Every Square Matrix Can be uniquely Expressed as the sum of a Hermitian and a Skew-Hermitian Matrix

Solution - Let A be any square Matrix and let A^* be its Tranjugate

then $A = \frac{A^*+A}{2} + \frac{A-A^*}{2}$

$= X + Y$

Where $X = \frac{A+A^*}{2}$ $Y = \frac{A-A^*}{2}$

Now $X = \frac{A+A^*}{2} \Rightarrow X^* = \frac{1}{2}(A+A^*)^*$

$= \frac{1}{2}[A^* + (A^*)^*]$

$= \frac{1}{2}[A^* + A]$

$= \frac{A+A^*}{2} = X$

$\Rightarrow X$ is Hermitian Matrix

$$Y = \frac{1}{2}[A - A^*]$$

$$Y^* = \frac{1}{2}[A - A^*]^*$$

$$= \frac{1}{2}[A^* - (A^*)^*]$$

$$= \frac{1}{2}[A^* - A]$$

$$= -\frac{1}{2}[A - A^*]$$

$$= -\frac{A - A^*}{2} = -Y$$

$\Rightarrow Y = -Y^* \Rightarrow Y$ is Skew Hermitian.

Thus we have

$A =$ Hermitian + Skew Hermitian

Further, suppose that A can also be expressed in other ways as

$A = W + Z$ — (I) where W is Hermitian, Z is Skew Hermitian

$\therefore A^* = (W+Z)^*$ i.e. $W = W^*$

$= W^* + Z^*$ $\quad Z = -Z^*$

$= W - Z$ ———— (II)

(I) + (II)

$2A^{\cancel{}} W \quad A + A^* = 2W$

$$W = \frac{A + A^*}{2} = X$$

(I) − (II)

$$A - A^* = 2Z$$

$$\Rightarrow \frac{A - A^*}{2} = Z$$

$$\Rightarrow Z = \frac{A - A^*}{2} = Y$$

This Representation of square Matrix A as the sum of Hermitian and a Skew Hermitian Matrix is unique.

(II) Show that $\bar{A}$ is Hermitian or Skew-Hermitian according as A is Hermitian or Skew-Hermitian

Proof :→ Let us suppose that A is Hermitian Hence

$$A^* = A \text{ ——— (1)}$$

then we claim that $\bar{A}$ is Hermitian.

$$(\bar{A})^* = \left(\overline{\bar{A}}\right)^T$$

$$= A^T$$

thus $(\bar{A})^* = A^T$ $\qquad \because A = A^*$

$$= (A^*)^T \qquad A^* = (\bar{A})^T$$

$$= [(\bar{A})^T]^T \qquad (A^T)^T = A$$

$$\therefore (\bar{A})^* = \bar{A} \Rightarrow \bar{A} \text{ is Hermitian Matrix}$$

Next, we suppose that A is Skew-Hermitian

$\therefore \quad A^* = -A$ ——— (ii)

then we have

$$(\bar{A})^* = [(\bar{A})]^T$$
$$= A^T$$
$$= [-A^*]^T$$
$$= -[A^*]^T$$
$$= -[(\bar{A})^T]^T$$
$$= -\bar{A}$$

$\Rightarrow (\bar{A})^* = -\bar{A} \Rightarrow \bar{A}$ is Skew Hermitian.

Show that Every Square Matrix A Can be uniquely expressed as $X+iY$ where X and Y are Hermitian Matrix

We know that Y is Hermitian iff iY is Skew Hermitian

Now we want to show that A Can be Expressed as the Sum of X and iY where X is Hermitian

Let $x = \frac{1}{2}[A + A^*]$ and $y = \frac{1}{2i}[A - A^*]$

then $x + iy = \frac{1}{2}[A + A^*] + \frac{i1}{2i}[A - A^*]$

$= \frac{1}{2}(A + A^*] + \frac{i}{2}(A - A^*]$

$= A$

$\therefore A = x + iy$

Now we have $x = \frac{1}{2}(A + A^*)$

$\Rightarrow x^* = \frac{1}{2}[A + A^*]^*$

$= \frac{1}{2}[A^* + (A^*)^*]$

$= \frac{1}{2}[A^* + A]$

$= \frac{1}{2}[A + A^*]$

$\because x = x^* \Rightarrow x$ is Hermitian Matrix

Also $y = \frac{1}{2i}[A - A^*]$

$\Rightarrow y^* = \left[\overline{\frac{1}{2i}(A - A^*)}\right]^T$

$= \left[\overline{\frac{1}{2i}}\,\overline{(A - A^*)}\right]^T$

$= \frac{1}{\overline{2i}}\left[\overline{A - A^*}\right]^T = -\frac{1}{2i}[A^* - A]$

$= -\frac{1}{2i}[-](A - A^*] = \frac{1}{2i}[A - A^*] = y$

thus $Y^* = Y \Rightarrow Y$ is also a hermitian Matrix.

thus
We came the conclusion that A can be expressed in the form of (1)

i.e $A = X + iY$ where X and Y are Hermitian Matrix

Now we shall show that the expression for A given by (1) is unique.
If possible, let A can be expressed as $A = W + iZ$ —— (1)
where W and Z both are Hermitian

i.e $W^* = W$ and $Z^* = Z$

$A = W + iZ \Rightarrow A^* = (W + iZ)^*$
$= W^* + (iZ)^*$
$= W^* - iZ^*$

$\Rightarrow A^* = W^* - iZ^*$ —— (11)

(1) + (11) $A + A^* = 2W$
$W = \frac{A + A^*}{2} = X$

(1) − (11) $A - A^* = 2iZ$, $Z = \frac{1}{2i}(A - A^*) = Y$

thus expression is unique