Coordinate Systems and Transformation

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or nonorthogonal.

Orthogonal system: An orthogonal system is one in which the coordinates are mutually perpendicular.

Non orthogonal systems are hard to work with, and they are of little or no practical use.

Examples of orthogonal coordinate systems include the Cartesian (or rectangular), the circular cylindrical, the spherical, the elliptic cylindrical, the parabolic cylindrical, the conical, the prolate spheroidal, the oblate spheroidal, and the ellipsoidal.

A considerable amount of work and time may be saved by choosing a coordinate system that best fits a given problem.

Cartesian coordinates (x, y, z)

A vector $\vec{A}$ in cartesian coordinates can be written as

$$(A_x, A_y, A_z) \text{ or } A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are unit vector along the x, y and z direction

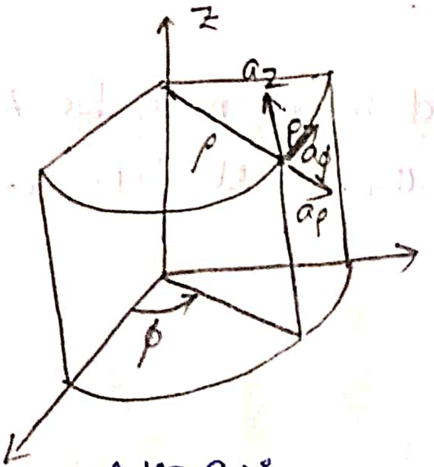
A point P can be represented as (x, y, z).

The range of the coordinate variable x, y and z are

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

Circular cylindrical coordinates (ρ, ϕ, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical coordinates



~~The range of the vari~~ ρ

A point P in cylindrical coordinates is represented as (ρ, ϕ, z).

ρ is the radius of the cylinder passing through P or the radial distance from the z-axis:

ϕ is called the azimuthal angle, is measured from the x-axis in the x-y plane

and z is the same as in the cartesian system.

The range of the variables are

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

A vector $\vec{A}$ in cylindrical coordinates can be written as

(A_ρ, A_ϕ, A_z) or ~~$A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$~~

$$A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$$

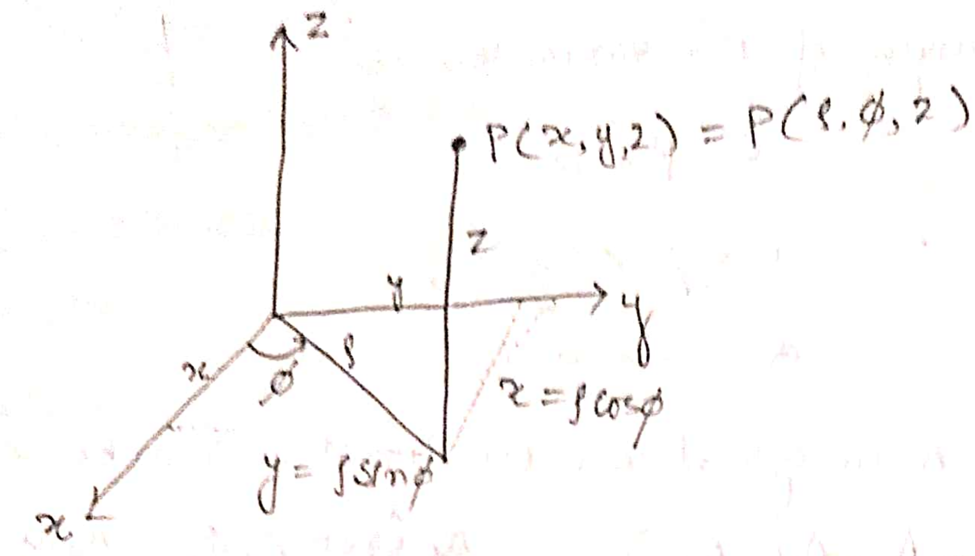
$\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$ are unit vectors in the ρ-, ϕ- and z-directions.

Magnitude of $\vec{A} = |\vec{A}| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$

Notice that unit vectors $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$ are mutually perpendicular because our coordinates system is orthogonal. Thus,

$$\left.\begin{aligned} \hat{\rho}\cdot\hat{\rho} = \hat{\phi}\cdot\hat{\phi} = \hat{z}\cdot\hat{z} = 1 \\ \hat{\rho}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{z} = \hat{z}\cdot\hat{\rho} = 0 \end{aligned}\right] \text{dot product}$$

$$\left.\begin{aligned} \hat{\rho}\times\hat{\phi} = \hat{z} \\ \hat{\phi}\times\hat{z} = \hat{\rho} \\ \hat{z}\times\rho = \hat{z} \end{aligned}\right] \text{cross product}$$



The relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z)

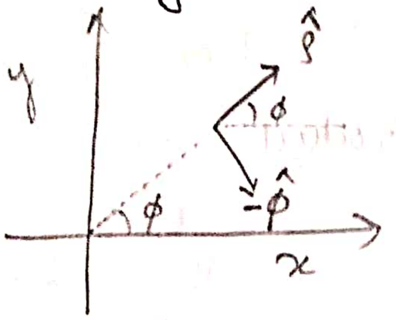
$$\therefore \quad \rho = \sqrt{x^2+y^2}$$

$$\phi = \tan^{-1} y/x$$

$$z = z$$

or $\quad x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

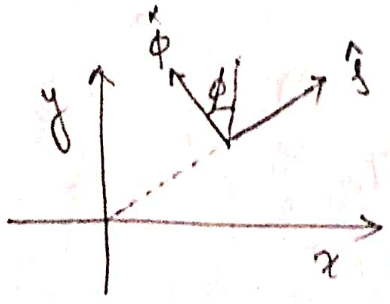
Finding relationship between $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{\rho}, \hat{\phi}, \hat{z})$



From figure we can see that $\hat{x} = \hat{\rho}\cos\phi - \sin\phi\,\hat{\phi}$

$$\hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$

$$\hat{z} = \hat{z}$$



or

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

So, in matrix form we write transformation of vector A from $(A_x, A_y A_z)$ to (A_ρ, A_ϕ, A_z) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse transformation

$(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

Differential element in cylindrical coordinates can be found as follows:

1. Differential displacement is given by

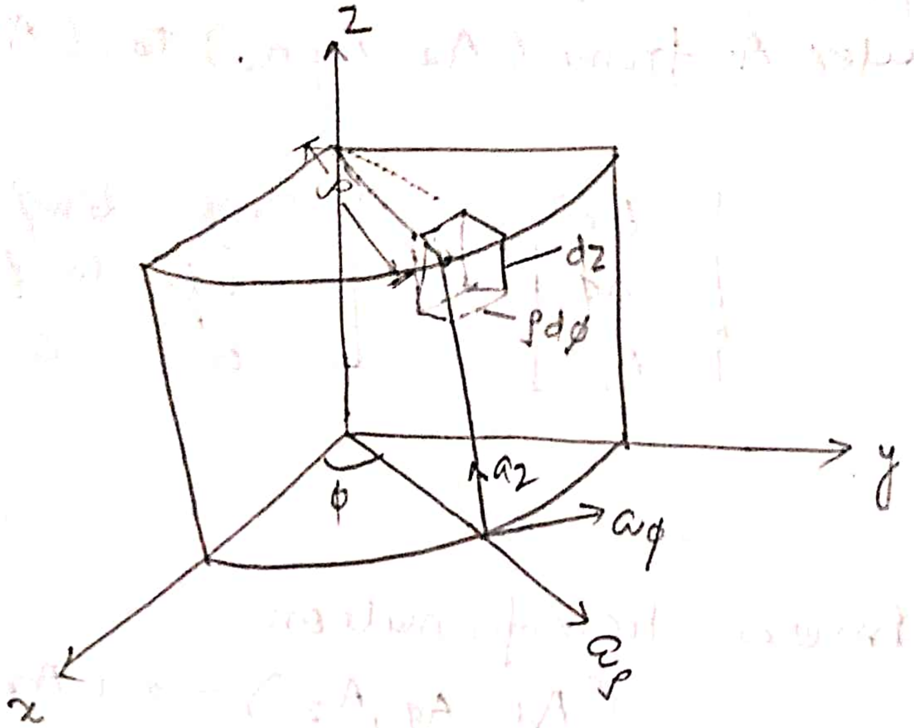
$$\vec{dl} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + z\,\hat{z}$$

2. Differential normal surface is given by

$$\vec{ds} = (\rho\, d\phi\, dz\,\hat{\rho}),\ (d\rho\, dz\,\hat{\phi}),\ (\rho\, d\rho\, d\phi\,\hat{z})$$

3. Differential volume is given by

$$dv = \rho\, d\rho\, d\phi\, dz$$


z
dz
ρdφ
ρ
φ
a_z
a_φ
a_ρ
y
x