[illegible]

* Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1+\cot x}$

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

Since $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Here $f(x) = \frac{\sin x}{\sin x + \cos x}$ $\quad a = \pi/2$

$\therefore \quad f(\pi/2 - x) = \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}$

$= \frac{\cos x}{\cos x + \sin x}$

Also $I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

$\therefore \; 2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} dx = [x$

$\therefore 2I = \pi/2 = \pi/2 - 0 = \pi/2$

$\Rightarrow I = \pi/4$

Problem :- Show that

$$\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi^2}{2ab}$$

Solution :- Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$f(x) = \frac{x}{a^2\cos^2 x + b^2\sin^2 x}$$

$$f(\pi - x) = \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x} - \frac{x}{a^2\cos^2 x + b^2\sin^2 x}$$

$$\therefore I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}$$

$$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \pi\int_0^{\pi} \frac{dx}{b^2\cos^2 x\left[\frac{a^2}{b^2} + \tan^2 x\right]}$$

$$= \frac{\pi}{b^2}\int_0^{\pi} \frac{\sec^2 x\,dx}{\tan^2 x + \frac{a^2}{b^2}}$$

$$= \frac{2\pi}{b^2}\int_0^{\pi/2} \frac{\sec^2 x\,dx}{\tan^2 x + \frac{a^2}{b^2}}$$

Put $\tan x = z \Rightarrow \sec^2 x\,dx = dz$

When $x = 0$ $z = 0$

When $x = \pi/2$ $z = \infty$

Where $K^2 = \frac{a^2}{b}$

$$2I = \frac{2\pi}{b^2}\int_0^{\infty} \frac{dz}{z^2 + K^2}$$

$$\therefore I = \frac{\pi}{b^2}\int_0^{\infty} \frac{dz}{z^2+K^2} \qquad \because K^2 = \frac{a^2}{b^2}$$

$$= \frac{\pi}{b^2}\cdot\frac{1}{K}\left[\tan^{-1}\frac{z}{K}\right]_0^{\infty} \qquad \therefore K = \frac{a}{b}, \quad \frac{1}{K} = \frac{b}{a}$$

$$= \frac{\pi}{b^2}\cdot\frac{b}{a}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{\pi}{ab}\cdot\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi^2}{2ab}$$

<u>Problem 3</u> Show that $\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2}$

$$I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx \qquad \because \cot^{-1}x = \tan^{-1}\frac{1}{x}$$

$$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}$$

$$= \int_0^1 \tan^{-1}\frac{1}{1+x(x-1)}\,dx$$

$$= \int_0^1 \tan^{-1}\frac{x-(x-1)}{1+x(x-1)}\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}(x-1)\,dx$$

$$\left[\int_0^1 \tan^{-1}(x-1)\,dx \qquad f(x) = x-1 \right.$$
$$\left. = \int_0^1 \tan^{-1}[(1-x)-1]\,dx \qquad f(1-x) = 1-x-1 \right]$$

$$\therefore I = \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}(x-1)\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}\{(1-x)-1\}\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}(-x)\,dx \qquad \because \tan^{-1}(-x) = -\tan^{-1}x$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx$$

$$= 2\int_0^1 \tan^{-1}x\,dx$$

Now integrating by parts

$$= 2\left[x\tan^{-1}x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 2\left[1\cdot\tan^{-1}1 - 0\right] - \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= 2\frac{\pi}{4} - \left[\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{2} - \left[\log(1+1) - \log(1+0)\right]$$

$$\log 2.$$

$$= \pi/2 - $$

Problem 4 Evaluate

$$\int_0^{\pi/2} \log \sin x \, dx$$

Let $I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$$= \int_0^{\pi/2} \log \cos x \, dx$$

$$\therefore \; 2I = \int_0^{\pi/2} [\log \sin x + \log \cos x] \, dx$$

$$= \int_0^{\pi/2} \log [\sin x \cos x] \, dx$$

$$= \int_0^{\pi/2} \log \frac{\sin 2x}{2} \, dx$$

$$= \int_0^{\pi/2} [\log \sin 2x - \log 2] \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, da - \log 2 \int_0^{\pi/2} dx$$

$$= I_1 - I_2$$

$I_1 = \int_0^{\pi/2} \log \sin 2x \, dx$ Putting $2x = z$

$2dx = dz$

When $x = 0$ $z = 0$

$x = \pi/2$ $z = \pi$

$$I_1 = \frac{1}{2}\int_0^{\pi} \log \sin z \, dz$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x \, dx$$

$$= I$$

$$I_2 = \log 2 \int_0^{\pi/2} dx$$

$$= \log 2 \, [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} \log 2$$

$$\therefore 2I = I - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}.$$

<u>Problem</u> Show that $\int_0^{\pi/2} \log(\tan x)\, dx = 0$

$$I = \int_0^{\pi/2} \log(\tan x)\, dx = \int_0^{\pi/2} \log(\cot x)\, dx$$

$$= \int_0^{\pi/2} \log(\tan x)^{-1}\, dx$$

$$= -\int_0^{\pi/2} \log(\tan x)\, dx$$

$$\Rightarrow 2I = 0$$

$$\therefore I = 0$$

Show that - $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$

$$\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \pi/8 \log 2.$$

Put - $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

When $x=0$ $\theta=0$

When $x=1$ $\theta=\pi/4$

$$\therefore I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\frac{2}{(1+\tan\theta)}\,d\theta = \log 2\int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \log 2\,[\theta]_0^{\pi/4} - I$$

$\therefore 2I = \frac{\pi}{4} \cdot \log 2$

$\therefore I = \frac{\pi}{8} \log 2$

Evaluate $\int_0^\infty \log\left(x + \frac{1}{x}\right) \frac{dx}{1+x^2}$

Let $I = \int_0^\infty \log\left(x+\frac{1}{x}\right) \cdot \frac{dx}{1+x^2}$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \cdot d\theta$. When $x = \infty$, $\theta = \tan^{-1}\infty = \pi/2$

When $x = 0$, $\theta = \tan^{-1} 0 = 0$

$= \int_0^{\pi/2} \log\left(\tan\theta + \frac{1}{\tan\theta}\right) \frac{\sec^2\theta \, d\theta}{\sec^2\theta}$

$= \int_0^{\pi/2} \log\left[\frac{1+\tan^2\theta}{\tan\theta}\right] d\theta$

$= \int_0^{\pi/2} \log\left[\frac{1}{\sin\theta\cos\theta}\right] d\theta$

$= -\int_0^{\pi/2} \log\sin\theta \, d\theta - \int_0^{\pi/2} \log\cos\theta \, d\theta$

$= -\int_0^{\pi/2} \log\sin\theta \, d\theta - \int_0^{\pi/2} \log\sin\theta \, d\theta$

$= -2\int_0^{\pi/2} \log\sin\theta \, d\theta = -2 \frac{\pi}{2} \log\frac{1}{2}$

$= \pi \log 2$