Name of the College :→ S.S. College, J. Bad

Deptt - Mathematics

Topic - Transpose and Tranjugate of a Matrix

Class - B.Sc I (Sub)

Time - 1.00 P.M To 1.45 P.M

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By : Samrendra Kumar

Study Material :→

The Matrix obtained by interchanging its rows and columns is called the Transpose of A where A is Given Matrix. Transpose of A is denoted by A^T

for if $A = [a_{ij}]_{m \times n}$

$$A^T = [a_{ij}]^T_{m \times n}$$

$$= [a_{ji}]_{n \times m}$$

Clearly

Transpose of Row Matrix is column Matrix

and Transpose of column Matrix is Row Matrix

$A = [1 \quad 2 \quad 3]_{1\times 3} \Rightarrow A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$ $\quad A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow A^T = [4 \quad 5 \quad 6]$

Properties of Matrix Transposition :→

If A and B are two Matrices of Compatible dimensions for the operations in which they appear and α is a scalar symmetric

Page 4

Page 2

theorem (i) $(A^T)^T = A$ (ii) $[A+B]^T = A^T + B^T$ (iii) $(\alpha A)^T = \alpha A^T$

(iv) $(AB)^T = B^T A^T$

show that $(AB)^T = B^T A^T$

$AB = [a_{ij}]_{m \times n} [b_{jk}]_{n \times p} = [c_{ik}]_{m \times p}$.

Where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$[AB]^T = [c_{ik}]^T_{m \times p}$

$= [c_{ki}]_{p \times m}$

We have $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$\Rightarrow c_{ki} = \sum_{j=1}^{n} \cancel{a}_{kj} b_{ji}$

thus $(AB)^T = \sum_{j=1}^{n} a_{kj} b_{ji}$ ——— (1)

Further, we have

$B^T A^T = [b_{jk}]^T_{n \times p} \cdot [a_{ij}]^T_{m \times n}$

$= [b_{jk}]_{p \times n} \; [a_{ji}]_{n \times m}$

$= \left[\sum_{j=1}^{n} b_{kj} \; a_{ji} \right]_{p \times m}$

$= \left[\sum_{j=1}^{n} a_{ji} \; b_{kj} \right]_{p \times m}$ ——— (2)

thus From (1) and (II) we have.

$(AB)^T = B^T A^T$

This Rule is known as Reversal Law for the Transpose.

Extension of the Reversal Law of Transposes :-

$$[ABC]^T = C^T B^T A^T$$

for $[ABC]^T = [(AB)C]^T$

$$= C^T (AB)^T$$

$$= C^T B^T A^T$$

Trace of a Matrix.

The sum of the main diagonal elements of a square Matrix $A = [a_{ij}]_{n\times n}$ is called the Trace of A and is called as "Spur". It is denoted by Tr A.

$$Tr A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Ex:-> Find the Trace of the Matrix.

$$A = \begin{bmatrix} t-2 & 3 & 1 \\ -4 & t+5 & t \\ -t & 2 & 2t-8 \end{bmatrix}_{3\times 3}$$

$\therefore$ Tr (A) = Sum of Diagonal elements

$= t-2 + t+5 + 2t-8$

$= 4t-5.$

Singular Matrix :->

A square Matrix $A = [a_{ij}]_{n\times n}$ is called Singular Matrix if $|A| = 0$

Non-Singular Matrix ->

If $A = [a_{ij}]_{n\times n}$ square Matrix and $|A| \neq 0$, Matrix is called Non-singular Matrix.

Symmetric Matrix →
If $A = [a_{ij}]_{n \times n}$ be a square matrix then A is Symmetric matrix if and only if $A = A^T$

Ex :→ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{3\times3} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{3\times3}$

Here $A = A^T \Rightarrow A$ is Symmetric Matrix

Skew Symmetric Matrix :→
A Square Matrix $A = [a_{ij}]_{n\times n}$ is said to be Skew Symmetric Matrix if and only if

$$A = -A^T$$

$$\Rightarrow a_{ij} = -a_{ji}$$

Ex :→ $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}_{3\times3}$

$A^T = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}_{3\times3}$

$\Rightarrow -A^T = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

thus $A = -A^T$

Show that Every Square Matrix can be uniquely Expressed as the sum of a Symmetric and Skew-Symmetric Matrix

Solution :→ Let A be a square matrix of order n

Then we have

$$A = \frac{A}{2} + \frac{A^T}{2} + \frac{A}{2} - \frac{A^T}{2}$$

$$= \frac{A+A^T}{2} + \frac{A-A^T}{2}$$

$\therefore$ $A = x + y$ where $x = \frac{A+A^T}{2}$
$y = \frac{A-A^T}{2}$ —— (1)

Now, $x = \frac{1}{2}[A + A^T]$

$$x^T = \frac{1}{2}[A + A^T]^T$$
$$= \frac{1}{2}[A^T + (A^T)^T]$$
$$= \frac{1}{2}[A^T + A]$$
$$= \frac{1}{2}[A + A^T]$$
$$= x$$

thus we have $x^T = x \Rightarrow \frac{A+A^T}{2}$ is Symmetric

and $y = \frac{1}{2}[A - A^T]$

$$y^T = \frac{1}{2}[A - A^T]^T$$
$$= \frac{1}{2}[A^T - (A^T)^T]$$
$$= \frac{1}{2}[A^T - A]$$
$$= -\frac{1}{2}[A - A^T]$$
$$= -y$$

where $Y^T = -Y \Rightarrow Y = -Y^T \Rightarrow Y$ is skew symmetric Matrix

Thus From (1)

A = Symmetric + Skew Symmetric

Now it remains to show that such a representation is unique.

If possible, say

$A = P + Q$ — (2)

where P = Symmetric

Q = Skew Symmetric

$\therefore A^T = (P+Q)^T$

$= P^T + Q^T$ Since $P = P^T$ as P is symmetric

$= P - Q$ — (3) $Q^T = -Q$ as Q is skew symmetric

(2)+(3)

$\frac{A + A^T}{2} = P = X$

(2) − (3) $\frac{A - A^T}{2} = Q = Y$

Hence the representation is unique

Complex Matrix :→

A Matrix is said to be complex, if all its elements are complex No and is usually denoted by z.

Any complex matrix can always be put into the form $P + iQ$ where P and Q are real matrix

Ex :→

1. $\begin{bmatrix} 2-7i & 3+4i \\ 7+5i & 6+8i \end{bmatrix}$

2. $\begin{bmatrix} 3+7i & -5 \\ 4 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 5i & 2+3i \\ -2+3i & 0 \end{bmatrix}$

These are the Example of complex Matrices.

Now $z = \begin{bmatrix} 3+7i & -5 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 3+7i & -5+i0 \\ 4+i0 & 7+i0 \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 3 & -5 \\ 4 & 7 \end{bmatrix}}_{=P} + i \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}}_{+iQ}$

Conjugate of A Matrix

A complex matrix $\overline{Z}$ is called conjugate of a matrix Z if it is obtained by replacing each element of Z by its conjugate.

Ex:→ $Z = \begin{bmatrix} 2-7i & 3+4i \\ 7+5i & 6+8i \end{bmatrix}$

$$\overline{Z} = \begin{bmatrix} 2+7i & 3-4i \\ 7-5i & 6-8i \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 7 & 6 \end{bmatrix} + i \begin{bmatrix} 7 & -4 \\ -5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 7 & 0 \end{bmatrix} - i \begin{bmatrix} -7 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= P - iQ.$$

Thus we have if $Z = P + iQ$

$$\Rightarrow \overline{Z} = P - iQ$$

$$\Rightarrow \overline{(\overline{Z})} = P + iQ = Z.$$

Observation :→ (i) $\overline{(\overline{A})} = A$

$\overline{\alpha A} = \overline{\alpha}\,\overline{A}$

$\overline{(A+B)} = \overline{A} + \overline{B}$

$\overline{AB} = \overline{A} \cdot \overline{B}$

Transposed Conjugate of a Matrix :→

The Transpose of $\bar{A}$ denoted by $(\bar{A})^T$ or A^* or A^H.

is referred to as the Transposed Conjugate of A or Tranjugate of A.

Or the Transpose of the Conjugate of Matrix A is called Transposed Conjugate of A.

Ex :→

$$A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}_{3\times 3}$$

$$\cancel{A^T} = \begin{bmatrix} \cancel{1-2i} & \cancel{2+3i} & \cancel{3-4i} \\ \cancel{4+5i} & \cancel{5-6i} & \cancel{6+7i} \\ \cancel{8} & \cancel{7-8i} & \cancel{7} \end{bmatrix}_{3\times 3}$$

$$A^T = \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$$

$$(\overline{A^T}) = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix} = A^*$$

Show that-

If A^* be Transjugate of A;
then $(A^*)^* = A$.

Let A^* be the Transjugate of A.

$$A^* = \overline{(A^T)}$$

$$(A^*)^* = (\overline{A^*})^T$$

$$= \left[\overline{\{(A^T)}\right]^T$$

$$= [A^T]^T$$

$$= A.$$

$\therefore (A^*)^* = A.$

2. $(AB)^* = B^* A^*$

Solution- $(AB)^* = \left[\overline{(AB)}\right]^T$

$$= (\bar{A}\,\bar{B})^T$$

$$= (\bar{B})^T (\bar{A})^T$$

$$= B^* A^*.$$