Name of College - S.S. College, J-Bad
Deptt - Mathematics
Topic - Reduction formulae
Class - B.Sc I (Hons)
Time - 10:15 A.M to 11:00 A.M
11:00 A.M to 11:45 A.M
Date - 16-09-2020
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Reduction Formulae :→

A formula connecting one integral with another in which the integral is of the same type but of lower degree or order, is called the reduction formula.

In most cases the reduction formula is obtained by the process of integration by parts but in some cases the method of differentiation are adopted.

Q. Obtain the Reduction formula for
(i) $\int \sin^n x \, dx$ (ii) $\int \cos^n x \, dx$
Where n is a +ve integer

Reduction formula for $\int \sin^n x \, dx$

Let $I_n = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \sin x \, dx$

Integrating by parts

$I_n = \sin^{n-1} x \int \sin x \, dx - \int \left[\int \sin x \, dx \cdot \frac{d}{dx} \sin^{n-1} x \right] dx$

$= \sin^{n-1} x (-\cos x) - \int -(\cos x)(n-1) \sin^{n-2} x \cos x \, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int [\sin^{n-2} x - \sin^n x] \, dx$

$= -\sin^{n-1} x \cos x + n-1 \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$

or $I_n = -\sin^{n-1} x \cos x + n-1 \int \sin^{n-2} x \, dx - (n-1) I_n$

$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

$\therefore I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$

Which is the required reduction formula

Q. Reduction formula for $\int \cos^n x \, dx$

Let $I_n = \int \cos^n x \, dx$

$$= \int \cos^{n-1} x \cdot \cos x \, dx$$

Integrating by parts.

$$I_n = \cos^{n-1} x \int \cos x \, dx - \int \left[\int \cos x \, dx \cdot \frac{d \cos^{n-1} x}{dx}\right] dx$$

$$= \cos^{n-1} x \cdot \sin x + \int \sin x \,(n-1) \cos^{n-2} x \, \sin x \, dx$$

$$= \cos^{n-1} x \sin x + \int \cos^{n-2} x \,(1 - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Obtain the Reduction formula for

$$\int \tan^n x \, dx$$

Let $I_n = \int \tan^n x \, dx$

$$= \int \tan^{n-2} x \,(\sec^2 x - 1)\, dx$$

Reduction formula for

$\int \sec^n x \, dx$

Let $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2} x \sec^2 x \, dx$

$$I_n = \int \sec^{n-2} x \sec^2 x \, dx$$

Integrating by parts.

$$I_n = \sec^{n-2} x \int \sec^2 x \, dx - \int \left[\int \sec^2 x \, dx \cdot \frac{d \sec^{n-2} x}{dx}\right] dx$$

$$= \sec^{n-2} x \cdot \tan x - \int \tan x \, (n-2) \sec^{n-3} x \cdot \sec x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int [\sec^n x - \sec^{n-2} x] \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$(1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Reduction formula for $\int \sin^m x \cos^n x \, dx$

Where m and n are +ve integers

We take

$P = \sin^{\lambda+1} x \cos^{\mu+1} x$ where λ and μ are smaller indices of $\sin x$ and $\cos x$ respectively in the two integrals to be connected

then we find $\frac{dP}{dx}$ and rearrange it as a linear function of the two integrals to be connected. Then we integrate both sides and solve for the required integrals by transposition.

(1) Obtain the Reduction formula for $\int \sin^m x \cos^n x \, dx$ by connecting it with (I) $\int \sin^{m-2} x \cos^n x \, dx$

(II) $\int \sin^m x \cos^{n-2} x \, dx$

In the two integrals

$$\int \sin^m x \cos^n x \, dx \quad \text{and}$$

$$\int \sin^{m-2} x \cos^n x \, dx$$

the smaller index of $\sin x$ is $m-2$ and that of $\cos x$ is n.

$$\therefore \lambda = m-2, \quad \mu = n$$

So, here $P = \sin^{\lambda+1} x \cos^{\mu+1} x = \sin^{m-1} x \cos^{n+1} x$

$$\therefore \frac{dP}{dx} = (m-1)\sin^{m-2} x \cos x \cos^{n+1} x + \sin^{m-1} x \,(n+1)\cos^n x\,(-\sin x)$$

$$= (m-1)\sin^{m-2} x \cos^n x (1-\sin^2 x) - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\left[\sin^{m-2} x \cos^n x - \sin^m x \cos^n x\right] - (n+1)\left[\sin^m x \cos^n x\right]$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m-1)\sin^m x \cos^n x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m x \cos^n x$$

$$\frac{dP}{dx} = (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m x \cos^n x$$

Integrating

$$P = (m-1)\int \sin^{m-2} x \cos^n x \, dx - (m+n)\int \sin^m x \cos^n x \, dx$$

$$\therefore \int \sin^m x \cos^n x \, dx = -\frac{P}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$$

i.e. $$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

$$(m+n) I_{m,n} = (m-1) I_{m-2,n} - \sin^{m-1} x \cos^{n+1} x$$