Name of College - S.S. College, J Bad
Deptt - Mathematics
Topic - problem based on Hyperbolic Function (Contd)
Date - 09-04-2021
Time - 10 A.M To 11.30 A.M
Class - B.Sc I (Hons)
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Problem :→ If $\sin(\theta+i\varphi) = \tan\alpha + i\sec\alpha$ show that $\cos 2\theta \cosh 2\varphi = 3$

Solution → By the question, $\sin(\theta+i\varphi) = \tan\alpha + i\sec\alpha$

$\Rightarrow \sin\theta\cos i\varphi + \cos\theta\sin i\varphi = \tan\alpha + i\sec\alpha$

$\left[\because \cos i\varphi = \cosh\varphi, \quad \sin i\varphi = i\sinh\varphi\right]$

$\Rightarrow \sin\theta\cosh\varphi + i\cos\theta\sinh\varphi = \tan\alpha + i\sec\alpha$

$\left[z_1 = z_2 \Rightarrow Re(z_1) = Re(z_2), \quad Im(z_1) = Im(z_2) \right]$

$\Rightarrow \sin\theta\cosh\varphi = \tan\alpha$ — (i)

$\cos\theta\sinh\varphi = \sec\alpha$ — (ii)

$\because \sec^2\alpha - \tan^2\alpha = 1$

using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\Rightarrow \cos^2\theta\sinh^2\varphi - \sin^2\theta\cosh^2\varphi = 1$

$\Rightarrow \frac{1+\cos 2\theta}{2}\sinh^2\varphi - \frac{1-\cos 2\theta}{2}\cosh^2\varphi = 1$

$\Rightarrow (1+\cos 2\theta)\sinh^2\varphi - (1-\cos 2\theta)\cosh^2\varphi = 2$

$\Rightarrow \sinh^2\varphi + \cos 2\theta\sinh^2\varphi - \cosh^2\varphi + \cos 2\theta\cosh^2\varphi = 2$

$$\Rightarrow \cos 2\theta\left[\sinh^2\varphi + \cosh^2\varphi\right] - \left[\cosh^2\varphi - \sinh^2\varphi\right] = 2$$

$$\left[\cosh^2\varphi + \sinh^2\varphi = \cosh 2\varphi \qquad \cosh^2\varphi - \sinh^2\varphi = 1\right]$$

$$\Rightarrow \cos 2\theta \cosh 2\varphi - 1 = 2$$

$$\Rightarrow \cos 2\theta \cosh 2\varphi = 3.$$

Problem :→ If. $\cos(x+iy) = \cos\alpha + i\sin\alpha$

Prove that

$$\cosh 2y + \cos 2x = 2$$

Solution :→ By the question,

$$\cos(x+iy) = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \cos x \cos iy - \sin x \sin iy = \cos\alpha + i\sin\alpha$$

$$\left[\text{use } \cos iy = \cosh y \quad \sin iy = i\sinh y\right].$$

$$\cos x \cosh y - i\sin x \sinh y = \cos\alpha + i\sin\alpha.$$

$$\left[\text{Since } z_1 = z_2 \Leftrightarrow \begin{array}{l} Re(z_1) = Re(z_2) \\ Im(z_1) = Im(z_2) \end{array}\right]$$

$$\Rightarrow \cos x \cosh y = \cos\alpha \quad \text{--- (i)}$$

$$-\sin x \sinh y = \sin\alpha \quad \text{--- (ii)}$$

Now. $\cos^2\alpha + \sin^2\alpha = 1$

$$\Rightarrow \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = 1$$

using

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cosh^2\theta = \frac{1}{2}(1+\cosh 2\theta)$

$\sinh^2\theta = \frac{1}{2}\left[\cosh 2\theta - 1\right]$

$$\Rightarrow \frac{1+\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1-\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2} = 1$$

$$\Rightarrow (1+\cos 2x)(1+\cosh 2y) + (1-\cos 2x)(\cosh 2y - 1) = 4$$

$\Rightarrow 1 + \cosh 2y + \cos 2x + \cos 2x \cosh 2y + \cosh 2y - 1$
$- \cos 2x \cosh 2y + \cos 2x = 4$

$\Rightarrow 2\cosh 2y + 2\cos 2x = 4$
$\Rightarrow \cos 2x + \cosh 2y = 2.$

Problem If $\cos(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$
prove that-
$$\phi = \frac{1}{2}\log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}$$

Solution :→ By the question
$$\cos(\theta + i\phi) = \rho[\cos\alpha + i\sin\alpha]$$

[In this problem we use the following
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos i\theta = \cosh\theta. \quad \sin i\theta = i\sinh\theta.$
$z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2)$
$Im(z_1) = Im(z_2)$]

$\Rightarrow \cos\theta\cos i\phi - \sin\theta\sin i\phi = \rho[\cos\alpha + i\sin\alpha]$
$\Rightarrow \cos\theta\cosh\phi - i\sin\theta\sinh\phi = \rho[\cos\alpha + i\sin\alpha]$

$\Rightarrow \cos\theta\cosh\phi = \rho\cos\alpha$ —— (1)
$-\sin\theta\sinh\phi = \rho\sin\alpha$ —— (11)
(1) ÷ (11)

$$\Rightarrow \frac{\cos\theta\cosh\phi}{\sin\theta\sinh\phi} = -\frac{\cos\alpha}{\sin\alpha}$$

$$\Rightarrow \frac{\cosh\phi}{\sinh\phi} = -\frac{\sin\theta\cos\alpha}{\cos\theta\sin\alpha}$$

$$\Rightarrow \tanh\phi = -\frac{\cos\theta\sin\alpha}{\sin\theta\cos\alpha}$$

[using $\tanh\theta = \dfrac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$]

$$\Rightarrow \frac{e^{\varphi} - e^{-\varphi}}{e^{\varphi} + e^{-\varphi}} = -\frac{\cos\theta \sin\alpha}{\sin\theta \cos\alpha}$$

$$\Rightarrow \frac{e^{-\varphi} - e^{\varphi}}{e^{-\varphi} + e^{\varphi}} = \frac{\cos\theta \sin\alpha}{\sin\theta \cos\alpha}$$

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$$\Rightarrow \frac{e^{-\varphi} + e^{\varphi}}{e^{-\varphi} - e^{\varphi}} = \frac{\sin\theta \cos\alpha}{\cos\theta \sin\alpha}$$

$$\Rightarrow \frac{e^{-\varphi}}{e^{\varphi}} = \frac{\sin\theta \cos\alpha + \cos\theta \sin\alpha}{\sin\theta \cos\alpha - \cos\theta \sin\alpha}$$ [using Componendo and Dividendo Rule].

$$\frac{e^{-\varphi}}{e^{\varphi}} = \frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}$$

$$\Rightarrow \frac{1}{e^{2\varphi}} = \frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}$$

$$\Rightarrow e^{2\varphi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

Taking log on both side.

$$2\varphi \log e = \log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}. \qquad \because \log e = 1$$

$$\Rightarrow 2\varphi = \log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}$$

$$\therefore \varphi = \frac{1}{2}\log\left\{\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right\}.$$ proved

Problem :→ If $\log \sin(\theta + i\varphi) = \alpha + i\beta$; show that

$$2\cos 2\theta = e^{2\varphi} + e^{-2\varphi} - 4e^{2\alpha}$$

Solution :→ By the question.

$$\log \sin(\theta + i\varphi) = \alpha + i\beta$$

$$\Rightarrow \log[\sin\theta \cos i\varphi + \cos\theta \sin i\varphi] = \alpha + i\beta$$

$$\Rightarrow \log[\sin\theta \cosh\varphi + i\cos\theta \sinh\varphi] = \alpha + i\beta.$$

$$\Rightarrow \log[x + iy] = \alpha + i\beta.$$

where $x = \sin\theta \cosh\varphi$.

$y = \cos\theta \sinh\varphi$.

$$\left[\text{Since } \log(m + in) = \frac{1}{2}\log(m^2 + n^2) + i\tan^{-1}\frac{n}{m}. \right]$$

$$\Rightarrow \frac{1}{2}\log[\sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi] + i\tan^{-1}\frac{\cos\theta \sinh\varphi}{\sin\theta \cosh\varphi} = \alpha + i\beta.$$

$$\Rightarrow \frac{1}{2}\log[\sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi] + i\tan^{-1}[\cot\theta \tanh\varphi] = \alpha + i\beta.$$

$$\Rightarrow \frac{1}{2}\log[\sin^2\theta \cdot (1 + \sinh^2\varphi) + (1 - \sin^2\theta)\sinh^2\varphi] + i\tan^{-1}(\cot\theta \tanh\varphi) = \alpha + i\beta$$

$$\Rightarrow \frac{1}{2}\log[\sin^2\theta + \sinh^2\varphi] + i\tan^{-1}(\cot\theta \tanh\varphi] = \alpha + i\beta.$$

$$\Rightarrow \frac{1}{2}\log(\sin^2\theta + \sinh^2\varphi] = \alpha \quad \text{(i)} \qquad \tan^{-1}[\cot\theta \tanh\varphi] = \beta. \quad \text{(ii)}$$

From ①

$$\log\left[\sin^2\theta + \sinh^2\varphi\right] = 2\alpha$$

$$\Rightarrow \sin^2\theta + \sinh^2\varphi = e^{2\alpha}$$

$$\Rightarrow \frac{1-\cos 2\theta}{2} + \frac{\cosh 2\varphi - 1}{2} = e^{2\alpha}$$

$$\Rightarrow 1-\cos 2\theta + \cosh 2\varphi - 1 = 2e^{2\alpha}$$

$$\Rightarrow \cosh 2\varphi - \cos 2\theta = 2e^{2\alpha}$$

$$\Rightarrow \cos 2\theta = \cosh 2\varphi - 2e^{2\alpha}$$

$$\Rightarrow \cos 2\theta = \frac{e^{2\varphi} + e^{-2\varphi}}{2} - 2e^{2\alpha}$$

$$\Rightarrow 2\cos 2\theta = e^{2\varphi} + e^{-2\varphi} - 4e^{2\alpha}$$

Result.